5.12 Additional Problems

5.297 Region R is the rectangle bordered by the lines $x = 5$, $y = 3$, and the coordinate axes.

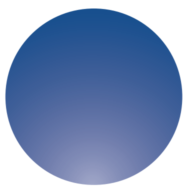
(a) If Region R has a density given by $\sigma = 10$, what is its total mass?

(b) If Region R has a density given by $\sigma = 10x$, what is its total mass?

(c) If Region R has a density given by $\sigma = 10y$, what is its total mass?

(d) If Region R has a density given by $\sigma = 10xy$, what is its total mass?

5.298 Uh-oh, my printer is running out of ink! I tried to print a solid blue circle, but the further the paper went in, the less ink I had, and the circle came out looking like this.



Glancing at it I notice that the ink is very thick on top, and gradually paler as the circle progresses. Measuring carefully I discover that the circle has a 5 inch radius and that the density of ink is given by approximately 2^h dots per square inch, where h is the height above the center of the circle in inches. (So at the top of the circle, $h = 5$; at the bottom, $h = -5$.)

(a) Calculate the density of dots at the very bottom of the circle. If the entire circle had that density, how many dots would it have?

(b) Draw a differential slice of the circle. Of course, your slice must have more or less a uniform density!

(c) Compute the area of your slice, as a function of h and dh only.

(d) Compute the number of dots in your slice.

(e) Set up an integral to represent the number of dots in the entire circle.

(f) Evaluate the integral. Give your answer rounded to the nearest integer.

5.299 Region R is bounded by the line $y = (L - x)/2$ and the x- and y-axes. Within this region is a metal whose density is given by ky^2. Both L and k are constants. Find the total mass of this metal.

For Problems 5.300–5.311 evaluate the given integrals. (Remember that unless there is a symbol you should be able to solve the problem using no technique more advanced than u-substitution or integration by parts. The first step is often choosing, or changing, the coordinate system.)

5.300 $\int_0^H \int_z^{2z} \int_{y-z}^{y+z} \left(x^2 - y^2\right)\, dx\, dy\, dz$

5.301 $\int_1^3 \int_{x^2}^9 x/(x^2 + y)^2\, dy\, dx$

5.302 $\int_{-5}^5 \int_{-\sqrt{25-y^2}}^0 (x^2 + y^2)^{3/2}\, dx\, dy$

5.303 $\int_{-H}^H \int_0^z \int_0^{\rho/z} d\phi\, d\rho\, dz$

5.304 $\int_{R_1}^{R_2} \int_0^{\cos^{-1}(r/R_2)} \int_0^\pi r^2 \sin\theta\, d\phi\, d\theta\, dr$

5.305 The integral of x^3 in the region in the first quadrant bounded by the lines $y = x$ and $x = 1/2$ and the curve $y = \sin(\pi x^2/2)$.

5.306 The integral of z^2 in the region bounded by the xy-plane, the cylinder $x^2 + y^2 = R^2$, and the paraboloid $z = H + \left(x^2 + y^2\right)/H$.

5.307 The integral of ρ (distance from the z-axis) over a sphere of radius R centered on the origin.

5.308 The integral of $r\cos^2\theta$ (where r and θ are the usual spherical coordinates) in the region bounded by the xy-plane and the cone $(z-1)^2 = x^2 + y^2$.

5.309 The integral of r (distance from the origin) over a cylinder of radius R and height H with its base centered on the origin.

5.310 The integral of the field $\vec{A} = 3\hat{i} + 5\hat{j}$ along…

(a) A straight line from $(0, 0)$ to $(6, 10)$.

(b) A straight line from $(0, 0)$ to $(10, -6)$.

(c) A straight line from $(0, 0)$ to $(-6, -10)$.

(d) A straight line from $(0, 0)$ to $(6, 0)$.

5.311 The integral of the field $\vec{B} = (x + y)\hat{i} + (xy)\hat{j}$ along…

(a) The curve $y = e^x$ from $x = 0$ to $x = 1$.

(b) The curve $x = \cos(2t)$, $y = 2t$ from $t = 0$ to $t = \pi$.

For Problems 5.312–5.317 indicate what region the integral is over. You can do this with a drawing, a description, or both, as long as you make it clear exactly what the region is. (Your description *cannot* simply describe the range of the coordinates. It has to use words, such as "a sphere of radius R centered on the origin" or "a cone with vertex at the origin, height H, and radius R on top.") You can tell what coordinate system each one is in by the names of the integration variables, e.g. x, y, and z for Cartesian coordinates and so on. Then evaluate the integral to find the volume of the region.

5.312 $\int_0^L \int_0^L \int_0^L dx\, dy\, dz$

5.313 $\int_{-R}^{R} \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{-\sqrt{R^2-y^2-z^2}}^{\sqrt{R^2-y^2-z^2}} dx\, dy\, dz$

5.314 $\int_0^H \int_0^R \int_{\pi/2}^{\pi} \rho\, d\phi\, d\rho\, dz$

5.315 $\int_{-H}^{H} \int_0^{R+kz^2} \int_0^{2\pi} \rho\, d\phi\, d\rho\, dz$

5.316 $\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, d\phi\, d\theta\, dr$

5.317 $\int_0^R \int_0^{\cos^{-1}(r/R)} \int_0^{2\pi} r^2 \sin\theta\, d\phi\, d\theta\, dr$

For Problems 5.318–5.322 you will be given the shape of an object. If no mass density is specified assume the object is uniform with total mass M. If no charge density is specified assume it is uniform with total density Q. Coordinate names such as ϕ and r have their usual meanings, and all other letters represent positive constants. For each object:

(a) If a mass density is specified, find the total mass of the object.

(b) If a charge density is specified, find the total charge of the object.

(c) Find the center of mass of the object (all components).

(d) Find the moment of inertia of the object about the z-axis.

(e) Find the electric potential produced by the object at the origin.

5.318 A triangle in the xy-plane is bounded by the y-axis, the line $y = 2$, and the line $y = 2x$. It has mass density axy and charge density bxy^2.

5.319 A sphere of radius R centered on the origin with mass density $ar^2 \sin\theta$ and charge density $b/(r + R)$. (The trig identity $\sin^2\theta = (1/2)[1 - \cos(2\theta)]$ may help with the integration here.)

5.320 The $x > 0$ half of a right circular cylinder of radius R centered on the z-axis and going from $z = -H$ to $z = H$. (For the potential set up the triple integral and evaluate two of the three integrations, but leave the answer as a single integral. All of the other integrals should be straightforward to evaluate.)

5.321 A four-sided pyramid whose base is a square going from $(-L, -L, 0)$ to $(L, L, 0)$ and whose vertex is at $(0, 0, H)$. The charge density is βr where r is (as usual) distance from the origin.

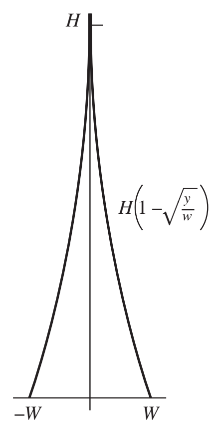
5.322 A sphere of radius R centered on the point $(0, 0, R)$.

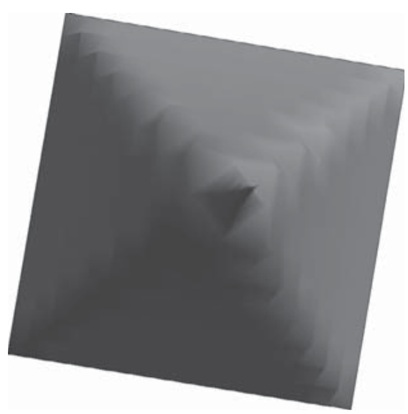
5.323 The Eiffel tower (shown below) has a square cross section at each height.

The surfaces on the sides are complicated, but can be well approximated by these two formulas.

$$\text{Front side: } z = H\left(1 - \sqrt{\frac{x}{W}}\right)$$

$$\text{Right side: } z = H\left(1 - \sqrt{\frac{y}{W}}\right)$$





The height H is about 320 m and W (which is half the width at the bottom) is about 62 m. Find the volume of the structure. (*Hint*: You'll make your life easier if you just use H and W until you get your final answer. Then you can easily check units, and then you can plug in numbers.)

5.324 The equations $x = v\cos u$, $y = v\sin u$, $z = u$ for $0 \le u \le 2\pi$ and $-1 \le v \le 1$ describe a "helicoid."

- **(a)** To see what this shape looks like on the xy-plane, substitute $u = 0$ into the equations for x and y. (Remember that u can take any value in $[0, 2\pi]$.) Draw the resulting shape on a set of xy-axes labeled $z = 0$.
- **(b)** Substitute $u = \pi/6$ into the equations for x and y. Draw the resulting shape on a set of xy-axes labeled $z = \pi/6$.
- **(c)** Do a similar drawing for representative u-values between 0 and 2π.
- **(d)** Describe and/or draw this helicoid.
- **(e)** Evaluate the surface integral of the function $\vec{f} = y\hat{i} + x\hat{j} - (x/\cos z)\hat{k}$ through this helicoid.

5.325 *[This problem depends on Problem 5.324.]* Problem 5.324 examined one particular helicoid. A more general formula is $x = av\cos u$, $y = av\sin u$, $z = bu$. How do the constants a and b change the helicoid?

5.326 The "impulse" exerted by a constant force on an object is equal to the force multiplied by the amount of time it acts. A particle experiences an exponentially growing force $F = ke^{t/\tau}$ from time $t = 0$ to time $t = 2\tau$. Find the total impulse on the particle.

5.327 The region bounded by the curves $x^2 + y^2 = 20$ and $x = y^2$ is filled with a charge density $\sigma = x(y + 1)$. Find the total charge in this region.

- **(a)** Sketch the two curves and the region between them.
- **(b)** Find the x- and y-coordinates of the points of intersection of the two curves and label them on your graph.
- **(c)** Your first impulse, looking at the region, might be to integrate over the top half only, and then double the answer to get the entire region. Explain why you cannot do so in this case.
- **(d)** The easiest (and therefore best) way to set up this problem is with horizontal slices. The lower limit on x is given by one simple formula for all slices, and the upper limit on x is given by one other simple formula for all slices. Write the double integral this way and evaluate it to find the total charge.
- **(e)** Vertical slices in this case require breaking the region up into two subregions. The left-hand region is bounded below by $y = -\sqrt{x}$ and above by $y = \sqrt{x}$; the right-hand region by different bounds. Set up the integrals to find and sum the total charges of these two regions separately.
- **(f)** Evaluate the integrals of your two regions and add them. You should of course get the same answer you found with horizontal slices!

5.328 The surface of a flat rock reflects different amounts of light in different places. The surface is a circle of radius R centered on the origin and the fraction of light reflected from each point is $k\rho$, where ρ is distance from the center of the circle. If light is shining uniformly on the entire surface, what fraction of that light is reflected?

5.329 A cylindrical tank of radius R and height H is filled with air and methane gas. The fraction of the total gas mixture that is methane gas is proportional to distance from the central axis. What fraction of the gas in the cylinder is methane? (The answer will contain an unknown constant of proportionality.)

5.330 A cylinder of radius 2 m and height 1.5 m is filled with a gas. The density of the gas drops exponentially with distance from the central axis, going from 2.5 kg/m^3 on the central axis to 1/5 that density at the outer edge.

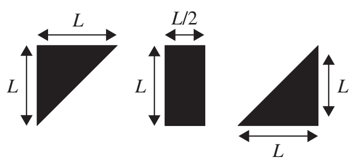
- **(a)** Write the density of the gas in Cartesian coordinates. If you use any letters (other than the coordinates and e) you should specify what their numerical values are.
- **(b)** Set up a Cartesian integral for the moment of inertia of the cylinder about its central axis.
- **(c)** Evaluate the integral numerically to find the moment of inertia of the gas.
- **(d)** Repeat Parts (a)–(c) in cylindrical coordinates. You should be able to verify that the integral appears different in the two coordinate systems, but gives the same answer.

5.331 The curve $x = \rho\cos(\omega t)$, $y = \rho\sin(\omega t)$, $z = H - vt$ is a vertical spiral. An object slides down this spiral from $t = 0$ until $t = H/v$ under the influence of gravity, which exerts a force of $\vec{F} = -mg\hat{k}$. Assume every letter in those formulas besides x, y, z, and t is a positive constant.

(a) Explain how you know that forces other than gravity must also be at work here, even though the problem does not mention them.

(b) How much work does gravity do on the object during this journey? (Even if you know the answer from physics, derive it with a line integral.)

5.332 The pressure of a fluid is the force per unit area that it exerts on its boundaries. In a tank of water the pressure increases linearly with depth. Suppose an aquarium tank has three viewing windows, two triangles and one square, as shown below. All three have the same area.[3]



(a) Which window has the greatest force on it from the water pressure? Which one has the least force on it? Explain how you know without doing any calculations.

(b) Now do the calculations to find the force on each window. Your answers will contain unspecified constants for the pressure at the top of the windows and the rate of increase of pressure with depth.

5.333 **Exploration: Gaussian Integrals**
The function $f(x) = e^{-x^2}$ is called a "Gaussian function." It has no indefinite integral in terms of elementary functions.[4] The mathematician Gauss figured out a clever trick, however, for evaluating the *definite* integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

We start by defining a constant that's equal to the number we are looking for: $S = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. (Not much progress so far, but we have to start somewhere.) Therefore, $S^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)$.

(a) The next step is to replace every x in the second integral with y. Explain briefly how we know that this does not change the answer.

(b) Combine the two integrals in your expression for S^2 into one double integral. (Section 3.2 explained "separable" integrals; run that process in reverse.)

(c) Use the laws of exponents to combine the two exponential functions into one.

(d) Looking at your limits of integration, what 2D region are you integrating over?

(e) Rewrite your double integral in polar coordinates. This involves rewriting the integrand in terms of ρ and ϕ and also replacing $dx\,dy$ with $\rho\,d\rho\,d\phi$. Use your answer to Part (d) to figure out the limits of integration in polar coordinates.

(f) You started with an integral that was impossible to evaluate. In polar coordinates you should now have one that is simple to evaluate. Do so and find the value of S^2. Then take the square root to get S, the number you were originally looking for.

(g) Use a computer to plot $\int_{-s}^{s} e^{-x^2}\,dx$ as a function of s, from $s = 0$ to $s = 10$. You should see the answer rapidly approach the value of S you computed in Part (f).

(h) What is $\int_0^{\infty} e^{-x^2}\,dx$? (Your answer should be exact, not approximate.)

(i) Explain why you cannot use this trick to calculate $\int_0^1 e^{-x^2}\,dx$.

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[3] We'd like to thank Drew Guswa for suggesting this problem to us.

[4] Mathematicians use this phrase a lot. What it really means is that there is no normal function whose derivative is e^{-x^2}, but it comes up a lot, so people created a function whose *definition* is $g(x) = \int_0^x e^{-t^2}\,dt$. So we can't technically say that the antiderivative of the Gaussian doesn't exist, only that it cannot be expressed in terms of other functions.