CHAPTER 5

Integrals in Two or More Dimensions (Online)

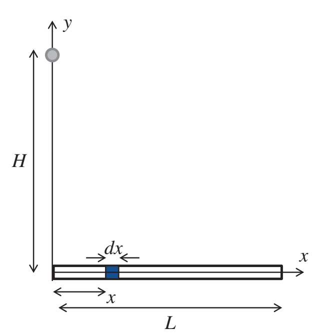
5.11 Special Application: Gravitational Forces

In the 17th century Isaac Newton proposed that every object exerts a gravitational force on every other object. The magnitude of this force is $Gm_1 m_2/r^2$ where G is a constant, m_1 and m_2 are the masses of the two objects, and r is the distance between them. The direction of the force on each object is directly toward the other one. Using this law, Newton was able to derive the orbits of the planets around the sun and the moon around the Earth, as long as he pretended that these bodies were all infinitesimal point masses. Unfortunately, they aren't. So Newton had to figure out how his new gravitational law would apply to an entire sphere.

You may be able to guess where this story is going … Newton broke the sphere up into lots of tiny pieces, calculated the force exerted by each piece, and then added them all up. In the limit where the pieces became infinitesimally small, this gave the exact answer. In short, he invented calculus.

This section will discuss the calculation of gravitational forces from extended objects. The procedure will be just like the other integration problems in this chapter with one new complication: force is a vector, so when we add the forces exerted by all the tiny pieces making up an object, we will need to add them component by component.[2] To illustrate how that works, we'll start with a much simpler problem, the gravitational field from a thin rod.

Say a uniform rod of mass M extends from the origin to the point $(L, 0)$. We want to find the gravitational force exerted by this rod on a mass m at the point $(0, H)$.



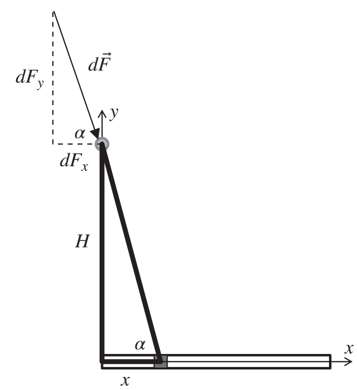
The picture above shows a slice at position x with thickness dx. The distance from the slice to the mass m is $r = \sqrt{x^2 + H^2}$. Because the rod has uniform density M/L the mass

[2]That's why people often calculate gravitational potential instead of gravitational force.

of the slice is $dm = (M/L)dx$, so the gravitational force exerted by that thin slice on the mass m is

$$dF = \frac{Gm\,dm}{r^2} = \frac{GmM}{L}\frac{dx}{x^2+H^2}$$

This is the *magnitude* of the force exerted by our dx slice. Integrating means adding up the contributions of all the slices, but to add the vectors $d\vec{F}$ we need to break them into components. The picture below shows the direction of $d\vec{F}$, with an angle labeled α, shown in two places on the figure.

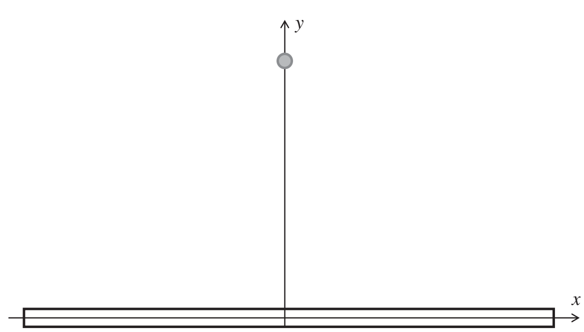


From this picture $dF_x = dF\cos\alpha = (x/r)dF$. The total x-component of the force of the rod on the mass is the sum of all the individual x-components:

$$F_x = \int_0^L \frac{Gm}{L}\frac{x}{\left(x^2+H^2\right)^{3/2}}dx = \frac{GmM}{L}\left[\frac{1}{H} - \frac{1}{\sqrt{L^2+H^2}}\right]$$

If you're wondering where that power of (3/2) came from, remember that dF has a $1/r^2$ in it and $\cos(\alpha)$ has a $(1/r)$ in it, so dF_x has $1/r^3$, which is $\left(x^2+y^2\right)^{-3/2}$. The terms in parentheses have units of one over distance, so the whole expression for F_x has units of GmM over distance squared, which is correct. We'll leave the y-component for you to work out in Problem 5.291.

In many situations you can save yourself some trouble by using the symmetry of the problem to eliminate some components. For example, suppose the rod had stretched from $(-L, 0)$ to $(L, 0)$ and the mass was still at $(0, H)$.



Just looking at the picture we can see that the mass is pulled equally to the left and right, so we can skip F_x and just calculate F_y.

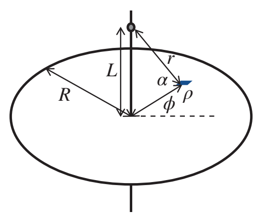
For 2D and 3D objects the process is the same, with a multiple integral to sum the force from all the differential boxes.

EXAMPLE Gravitational Force from a Circle

Question: A flat disk of radius R has density $\sigma = k\rho^2$, where ρ is distance from the center of the disk. Find the gravitational force exerted by the disk on a mass m along its central axis, a distance L from the center of the disk.

Answer: First we need to choose a coordinate system. Remember that the integration is just over the disk, so this will be a 2D integral. We choose our axes so that the disk is in the xy-plane and the z-axis goes through the center, and we choose polar coordinates.

The disk and mass are shown below, along with a small box at polar coordinates (ρ, ϕ). The mass of the small box is $dm = \sigma dA = (k\rho^2)(\rho d\rho d\phi)$ and the distance from the box to the mass is $r = \sqrt{\rho^2 + L^2}$.



By symmetry we know the force of the disk will point straight down, so we only need to find the z-component. The force points directly along the line labeled r, so $dF_z = -dF \sin\alpha = -dF(L/r)$, where α is the angle labeled in the picture. Putting everything together

$$dF_z = -\frac{Gmdm}{r^2}\frac{L}{r} = -Gmk\frac{\rho^3}{\left(\rho^2 + L^2\right)^{3/2}} d\rho d\phi$$

$$F_z = -Gmk \int_0^R \int_0^{2\pi} \frac{\rho^3}{\left(\rho^2 + L^2\right)^{3/2}} d\phi d\rho$$

The ϕ integral is trivial and just gives a factor of 2π. You can approach the ρ integral with a trig substitution, or just pop it into a computer, to get:

$$\vec{F} = -\frac{2\pi Gmk}{3}\left(2L^3 + \left(R^2 - 2L^2\right)\sqrt{R^2 + L^2}\right)\hat{k}$$

We'll leave it to you to use the density function $\sigma = k\rho^2$ to find the units of k and then check the units of this answer.

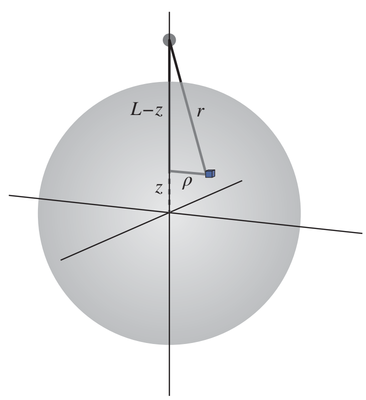
We are now at last ready to return to the problem that began, not only this section and this chapter, but in many ways the history of modern mathematics.

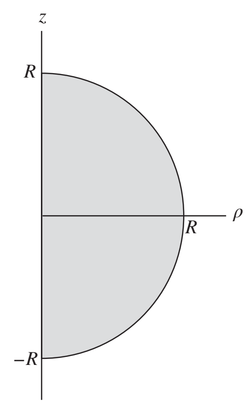
EXAMPLE

Newton's Problem, or The Gravitational Force from a Sphere

Question: Find the gravitational force exerted by a uniform sphere of mass M and radius R on a mass m at a distance L from the center of the sphere, where $L > R$.

Answer: For this problem it's not obvious what the right coordinate system is. The limits of integration are easy in spherical coordinates, but the distance from a box at coordinates (r, θ, ϕ) to the mass m is messy in this coordinate system. Conversely, that distance is easiest to express in Cartesian coordinates, but the limits of integration are messy. Cylindrical coordinates split the difference: medium messiness in both the distance formula and the limits of integration. We'll set it up it here in cylindrical coordinates, and you will do it in the problems in the other two coordinate systems.





We've defined the z-axis to go from the center of the sphere to the mass m, so the mass is at $(0, 0, L)$. The distance from a small box at (ρ, ϕ, z) to the mass is $r = \sqrt{\rho^2 + (L-z)^2}$. The density of the sphere is M/V where $V = (4/3)\pi R^3$, so the mass of the small box is $3M/\left(4\pi R^3\right) \rho d\rho d\phi dz$. By symmetry we only need the z-component of the force, which is given by $dF_z = -dF(L-z)/r$.

For the limits of integration, ϕ is easiest as usual: 0 to 2π. That leaves the 2D shape shown above, where the curved edge is the semicircle with the equation $\rho^2 + z^2 = R^2$. So ρ goes from 0 to $\sqrt{R^2 - z^2}$ and z from $-R$ to R. We now have all the pieces in place:

$$F_z = -\frac{3GmM}{4\pi R^3} \int_{-R}^{R} \int_0^{\sqrt{R^2-z^2}} \int_0^{2\pi} \frac{\rho(L-z)}{\left(\rho^2 + (L-z)^2\right)^{3/2}} \, d\phi \, d\rho \, dz$$

After you evaluate the ϕ and ρ integrals, you end up here:

$$\frac{3GMm}{2R^3} \int_{-R}^{R} \left(\frac{L-z}{\left[R^2 - z^2 + (L-z)^2\right]^{\frac{1}{2}}} - 1 \right) dz$$

You'll use integration by parts in Problem 5.293 to show that this comes out to $-GMm/L^2$. Of course you can also check this with a computer. (In later chapters you'll learn easier ways to calculate gravitational forces.)

5.11.1 Problems: Gravitational Forces

5.282 **Walk-Through: Finding Gravitational Force.** A uniform square of mass M extends from $(-L, -L)$ to (L, L). A point mass m is at position $(0, 2L)$.

(a) Draw the square and the point mass. Include in the square a differential box. Label the position (x, y) of the box and the distance r from the box to the mass m.

(b) Find the mass dm of the tiny box and the distance r from the box to the mass.

(c) Find the magnitude of the gravitational force dF exerted by the tiny box on the mass m.

(d) Explain how we can know that the total gravitational force of the square on the mass will be in the $-y$-direction.

(e) Calculate the y-component of the gravitational force dF that you found.

(f) Set up a double integral over the square to add up the contributions from all the differential boxes.

(g) Evaluate the double integral to find the total gravitational force $\vec{F}$ exerted by the square on the mass m. Express your answer in unit vector notation and check that it has correct units.

In Problems 5.283–5.289 find the gravitational force exerted by a uniform object of mass M with the specified shape on a point mass m at the specified location. Remember to use symmetry so you don't calculate more components of $\vec{F}$ than you need to.

5.283 A thin rod of length L on a mass a distance w from the end of the rod (along the same line as the rod).

5.284 A thin rod stretching from $(-L, 0)$ to $(L, 0)$ on a mass at point $(0, H)$.

5.285 A square in the xy-plane with corners at $(-L, -L, 0)$ and $(L, L, 0)$ on a mass at the point $(0, 0, H)$.

5.286 A uniform disk of radius R on a mass along its central axis at a distance H from the center of the disk.

5.287 A disk of radius $R = 0.5$ m and mass $M = 5$ kg on a 2 kg mass a distance R from the edge of the disk (but in the same plane).

5.288 A 5000 kg cube of edge length $L = 10$ m on a 1 kg mass attached to one of its corners.

5.289 A right circular cylinder of height H and radius R on a mass along its central axis at a distance L above the bottom of the cylinder. (Assume $L > H$ so the mass m is outside the cylinder.)

5.290 A thin rod of uniformly distributed mass M stretches from $(0, 0)$ to $(0, L)$. You are going to find its gravitational force on a mass m at a point (x, y) where x and y are both larger than L. (*Hint*: Be careful not to name a variable of integration y, since y already means one of the coordinates of the mass m in this problem.)

(a) Write integrals that represents the components of the force.

(b) Calculate the y-component of the force.

(c) Calculate the x-component of the force.

(d) Calculate the magnitude of the force and take its limit as $L \to 0$. Explain why your answer makes sense.

5.291 In the Explanation we calculated the x-component of the gravitational force exerted by a uniform rod of mass M from the origin to $(L, 0)$ on a mass m at $(0, H)$. Find the y-component of that force.

5.292 A thin rod goes from $(-1, 0)$ to $(1, 0)$ with density $\lambda = e^x$. Find the gravitational force of this rod on a rod of mass 1 at position $(0, 3)$. Assume all numbers are in SI units, and just leave G in your answer.

5.293 In the example "Newton's Problem" we set up an integral to calculate the gravitational force exerted by a uniform sphere, using cylindrical coordinates. In this problem you will finish that calculation, using integration by parts.

(a) The second term in parentheses is just the number 1. Evaluate that part of the integral.

(b) Focusing on the first term in the parentheses, the denominator contains $(L - z)^2$. Expand that out and simplify as much as possible. You should end up with a linear function of z under the square root.

(c) The numerator is $L - z$. Split this into two fractions and evaluate the part where the numerator is L using u-substitution. (Remember that $L > R$ so $\sqrt{(R-L)^2} = L - R$.)

(d) The one thing you have left to integrate is of the form $z/\sqrt{a+bz}$, where a and b are functions of R and L (and therefore constants). Integrate this using integration by parts with $u = z$ and $dv = dz/\sqrt{a+bz}$.

(e) Put all your answers together to find the gravitational field from a sphere.

5.294 In the example "Newton's Problem" we set up an integral to calculate the gravitational force exerted by a uniform sphere using cylindrical coordinates. Set up triple integrals to calculate the same quantity in Cartesian and spherical coordinates. (You do not need to evaluate the integrals. Presumably you know what the result would be.)

5.295 A short, straight wire of length ds carrying a current I produces the following magnetic field at a nearby point P.

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \vec{r}}{r^3}$$

Here $d\vec{s}$ is a vector of length ds that points in the direction of the current flow and $\vec{r}$ goes from the wire to P. We use r for the magnitude of $\vec{r}$. A circular wire of radius R is in the xy-plane, centered on the origin, carrying a counterclockwise current I. Find the magnetic field this wire produces at the point $(0, 0, H)$.

5.296 Exploration: Two Extended Objects
A uniform cube mass M_1 has two opposite corners at $(0, 0, 0)$ and (L, L, L). Another uniform cube of mass M_2 has opposite corners at $(0, 0, 2L)$ and $(L, L, 3L)$. In this problem you will find the gravitational force of cube 1 on cube 2.

(a) Draw the two cubes and draw a tiny box inside each one. Remember that you can't use the same label for two different variables, so you should call the position of one box (x, y, z) and the other one (x', y', z').

(b) Use Newton's law for the gravitational force between point particles to find the force exerted by the tiny box in cube 1 on the tiny box in cube 2.

By symmetry, the total force of cube 1 on cube 2 must be in the z-direction, so we will henceforth ignore the x- and y-components.

(c) Find the z-component of the force you found in Part (b).

(d) Set up, but do not yet evaluate, an integral for the z-component of the force of all of cube 1 on all of cube 2.

(e) That sixth-order integral is hard to evaluate, even for some computer algebra programs. As it stands now you can't evaluate it numerically because the limits have L in them. Set $L = 1$ and evaluate the integral numerically. The result for F_z should still include G, M_1, and M_2. Using that result and the requirement that the answer have units of force, find F_z in terms of G, M_1, M_2, and L.