Motivating Exercise for Special Functions and ODE Series Solutions The Circular Drum

In this exercise you will solve a partial differential equation. This process may be new to you or it may be review. One purpose of this exercise is to teach and/or remind of you this process. But focus your attention on the steps we skip—the times we just look up an answer and accept it instead of solving for it. The topic you are beginning to study, "special functions and series solutions," fills in those gaps.

A circular drumhead of radius a is allowed to vibrate. If the initial state of the drum has "azimuthal symmetry" (no ϕ dependence) then the drum will continue to have azimuthal symmetry over time. Under that circumstance the height z is a function of the polar distance ρ and the time t, and is governed by the following PDE (where v is a constant).

$$\frac{\partial^2 z}{\partial t^2} = v^2 \left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho} \right) \quad \text{the wave equation in polar coordinates with azimuthal symmetry} \tag{1}$$

The technique called "separation of variables" tells us to write the solution as the product of two different functions.

$$z(\rho, t) = R(\rho)T(t)$$

The next few steps—important in the study of PDEs but not relevant for our purpose here—take us to the two equations below. Note the introduction of a new constant k.

$$T''(t) = -k^2 v^2 T(t) \tag{2}$$

$$\rho^2 R''(\rho) + \rho R'(\rho) + k^2 \rho^2 R(\rho) = 0 \tag{3}$$

1. Find the general solution to Equation 2. You should be able to do this by inspection.

2. Write the general solution to Equation 3 by looking it up in a table[1] or using a computer. *Hint*: Some of the ODEs you might look up in a table are actually classes of ODEs identified by different values of a parameter p. Equation 3 is the $p = 0$ version of one such ODE.

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3. An "implicit boundary condition" that $R(\rho)$ must obey is that it must be finite at $\rho = 0$. Look up the basic properties of the functions you found in Part 2, and use this implicit boundary condition to set one of your two arbitrary constants to zero.

4. The other boundary condition is $R(a) = 0$ because the outer edge of the drum can't move up and down. Use that boundary condition to limit the possible values of k, once again looking up information about your function. There are infinitely many possible values for k so your answer will contain a new parameter n, where n can be any positive integer.

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[1] Our favorite table is Appendix J in *Mathematical Methods in Engineering and Physics* by Felder and Felder, but there are others online and in print.

This should all make sense if you're willing to simply accept the information you got from a computer or lookup table, but where did that information come from? The methods of "Power Series" and "Frobenius" can be used to find solutions to equations such as 3—solutions that are expressed in the form of series. Some solutions you find in that way, most notably "Bessel functions" and "Legendre polynomials," are worthy of special study.

There is another gap, maybe harder to see. At the $\rho = 0$ boundary we did not specify a value of R; we merely asserted that it must be finite. At the $\rho = a$ boundary we needed a specific value. "Sturm-Liouville theory" provides a rule for determining what kind of condition is needed at a boundary.

You're not done yet, though. You still need to combine these functions into a solution $z(\rho, t)$ and match initial conditions.

5. Multiply your solutions $R(\rho)$ and $T(t)$ to find a solution $z(\rho, t)$. You should be able to absorb the remaining arbitrary constants in $R(\rho)$ into the arbitrary constants in $T(t)$ to give you a solution $z(\rho, t)$ with two arbitrary constants. Don't write k in your answer; use the value of k that you found above.

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6. The solution you just found is called a "normal mode" of the PDE. There are infinitely many such normal modes, one for each value of n. Write the general solution $z(\rho, t)$ as a sum of all of these normal modes. Since the arbitrary constants can take different values for each n, write a subscript n on them.

7. For simplicity we'll use as one initial condition $\dot{z}(\rho, 0) = 0$. Plug this into your solution and show that one of the two arbitrary constants must be zero. This should leave you with an infinite series with just one undetermined set of constants B_n. (Of course you may have used a different letter, but we'll refer to it as B_n from here on.)

8. Our other initial condition will be an unspecified function $z(\rho, 0) = f(\rho)$. Plugging $t = 0$ into your solution for z write $f(\rho)$ as an infinite series.

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9. Use a table lookup one last time to find how to determine the coefficients of a series of the type you just wrote. Express B_n as an integral involving the unknown function $f(\rho)$.

Once again it all relied on looking up magic formulas. Can any initial condition $f(\rho)$ can be written in a series of this form? It turns out the functions that you found have a property called "completeness," which essentially means yes, any $f(\rho)$ can be written as a sum of these functions. Sturm-Liouville theory explains under what circumstances the solutions of a given ODE are complete in this sense, and also explains how to derive the formula you used for finding the coefficients B_n of such a series.