Discovery Exercise: Guess and Check, and Linear Superposition

If we ask you to find a solution to the equation

$$\frac{d^2u}{dx^2} - \left(\frac{1}{x} + 2x\right)\frac{du}{dx} - 8x^2u = 0 \tag{1}$$

you might quite reasonably have no idea how to begin. So we're going to tell you something else: there is a solution of the form $u(x) = e^{(something)x^2}$. Of course there is no obvious way you could have figured that out from scratch, but now that you know it, your job is to find the *something,* which is much more doable.

1. Test $u(x) = e^{x^2}$ in Equation 1 and show that it *doesn't* work. *Hint*: don't forget the product rule when calculating $u''(x)$!

One down, right? But rather than checking every possible value of that *something*, you can check all of them at once.

2. Try the guess $u(x) = e^{px^2}$ where p is a constant. Plug this guess into Equation 1 and simplify the result until you get a quadratic equation for p. *Hint*: see the hint above.

3. Solve that quadratic equation to find the two values of p for which your guess works. Write down the two solutions $u_1(x)$ and $u_2(x)$ that you have found.

   See Check Yourself # 6 in Appendix L

Note that p was not an *arbitrary constant* because the solution you tried only worked for certain values of p.

4. Verify that u_1 and u_2 are solutions to Equation 1.

5. Show that $4u_1(x)$ is also a solution.

6. Show that $Au_1(x) + Bu_2(x)$ is a solution for *any* constants A and B.