Computer Problems for Ordinary Differential Equations

1. **Compound Interest**

   Shannon put \$1000 into a bank account with 5% interest. If you solve the equation $dM/dt = M/20$ with initial condition $M(0) = 1000$ you get $M(t) = 1000e^{t/20}$. You can easily check that this causes her money to increase by more than 5% after a year because her balance keeps growing, and she immediately starts earning interest on that new balance. In this problem we take a numerical approach to the question: how much money does Shannon have after 20 years? In all cases, round off your answer to the nearest penny.

   (a) If Shannon's money is "compounded annually"—that is, if she receives exactly 5% of her balance once a year—then the formula for her balance is $M(t) = 1000 \times 1.05^t$. Compute her balance after 20 years.

   (b) If Shannon's money is "compounded monthly"—that is, if she receives 1/12 of 5% of her balance every month—then the formula for her balance is $M(t) = 1000 \times (1 + .05/12)^{12t}$. Compute her balance after 20 years.

   (c) If Shannon's money is "compounded daily"—that is, if she receives 1/365 of 5% of her balance every month—then the formula for her balance is $M(t) = 1000 \times (1 + .05/365)^{365t}$. (Ignore leap years.) Compute her balance after 20 years.

   (d) Many banks advertise that your money is "compounded continuously." Use a computer to take the limit of the above process as $n \to \infty$.

   (e) Plug $t = 20$ into the solution $M(t) = (\$1000)e^{t/20}$. Comparing the result to your answers above, what kind of compounding does this solution represent?

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

2. Consider the linear differential equation $dy/dx = 3 - y$.

   (a) For the function $y = e^{-x} + 3$, calculate dy/dx.

   (b) For the function $y = e^{-x} + 3$, calculate and simplify $3 - y$.

   (c) Your answers to parts (a) and (b) should have come out the same, demonstrating that $y = e^{-x} + 3$ is a solution to this differential equation. Using the same method, demonstrate that $y = 7e^{-x} + 3$ is *also* a solution to this differential equation.

   (d) Show that the function $y = Ce^{-x} + 3$ is a solution to this differential equation for any value of the constant C. (You do this the same way you did the previous problems: just let C work through the equations as a constant.)

   (e) What specific function does this general solution give you for the value $C = 0$? Show that this function is a solution to the differential equation.

   (f) Find the one and only function that satisfies both the differential equation $dy/dx = 3 - y$ and the initial condition $y(0) = 1$.

   (g) Show that the function $y = 3 - e^{10-x}$ is a solution to this differential equation. Then explain why this does not violate the rule that the solution in Part (d) is the general solution representing all possible solutions.

3. *[This problem depends on Problem 2.]* The general solution for the differential equation $dy/dx = 3 - y$ is $y = Ce^{-x} + 3$.

   (a) Graph the solutions for this equation for $C = -3$, $C = -2$, $C = -1$, $C = 0$, $C = 1$, $C = 2$, and $C = 3$. Limit your graphs to $x \geq 0$.

   (b) Discuss in words what all these functions have in common. How do they evolve as x increases? What is $\lim_{x\to\infty} y(x)$?

4. $dy/dx = 1 - y^2$
   (a) Create a slope field for all integer points $0 \le x \le 3$, $-3 \le y \le 3$.
   (b) Based on your slope field, describe the behavior of $y(x)$ if...
      i. $y(0) > 1$
      ii. $y(0) = 1$
      iii. $-1 < y(0) < 1$
      iv. $y(0) = -1$
      v. $y(0) < -1$
   (c) Solve the original equation to find a function $y(x)$. (The easiest way to do this by hand requires the technique of partial fractions for your integral. Alternatively you could set up the integrals by hand and then evaluate them with a computer.)
   (d) Show that your resulting $y(x)$ function is a valid solution to the original differential equation.
   (e) Find the particular solutions corresponding to the initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$. This will involve either finding the right value of the arbitrary constant or allowing it to approach ∞.
   (f) Plot the three particular solutions you just found. You should be able to see that they display the behaviors you described in Part (b).

In Problems 5–9 you will be given a differential equation, a set of initial conditions, and a final time. Solve the equation numerically, plot the solution, and find the value of $y(t_f)$.

5. $dy/dt = t^3 - y^3$, $y(0) = 1/10$, $t_f = 2$

6. $d^2y/dt^2 = -y^3$, $y(0) = 1/10$, $y'(0) = 0$, $t_f = 2$

7. $d^2y/dt^2 = -\sin(y)$, $y(0) = 0$, $y'(0) = 1$, $t_f = 2\pi$

8. $d^2y/dt^2 + (dy/dt)^2 + y = 1$, $y(0) = 0$, $y'(0) = 0$, $t_f = 5$

9. $d^3y/dt^3 = -y^2$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = .2$, $t_f = 2$

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10. For the differential equation $f''(x) = -(1/x)f(x)$ with boundary condition $f(0) = 0$ how many times does the solution cross the x-axis in the range $0 < x < 100$? Notice that $x = 0$ is not included in this range, so you shouldn't count the initial condition at the origin as one of the axis crossings. (*Hint*: Because this is a second order equation with only one boundary condition the solution will have an arbitrary constant in it. You should still be able to answer this question.)

11. For the differential equation $y''(x) + xy'(x) + y(x) = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, the solution rises to a maximum, then falls, and then decreases towards zero. Find the value of x where this solution reaches its maximum value.

12. For the differential equation $y''(x) + y'(x) + xy(x)$ with initial conditions $y(0) = 0$, $y'(0) = 1$, find the first positive value of x at which the solution $y(x)$ equals zero.

13. Important Quantity I follows the differential equation $dI/dt = I^4 - 6I^3 + 9I^2 - 4I$.

(a) Have a computer find the solutions to the equation $I^4 - 6I^3 + 9I^2 - 4I = 0$. Explain how you know that those solutions are the equilibrium values of I.

(b) Have a computer generate a slope field with a range of values for I that includes all the equilibrium values you found in Part (a).

(c) Using this slope field, predict the long-term behavior of I. Your answer will consist of several different statements of the form "If I starts in this range, then it will head toward…"

14. Find all the equilibrium values of the differential equation $dx/dt = 4x^4 - 4x^3 - 4x^2 + 4x$ and classify each one as stable, unstable, or neither.

15. Consider the differential equation $y'(t) = \sin y$.

(a) Have a computer solve this equation analytically.

(b) Based on your solution, what is $\lim_{t\to\infty} y(t)$ if $y(0) = \pi/2$?

(c) What are the equilibrium values for this equation? *Hint*: There are an infinite number of them.

(d) Draw a slope field for this equation. You can do this by hand or with a computer. Your graph should show at least three equilibrium values.

(e) Make sure your slope field confirms your answer to Part (b) and then use it to find $\lim_{t\to\infty} y(t)$ if $y(0) = -\pi/2$.

16. Consider $y'(t) = \sqrt{y - y^2}$

(a) Have a computer solve this analytically. Verify that the solution works.

(b) Draw a slope field for this equation for $0 \le y \le 1$. (Why did we have to restrict it to that range?)

(c) You may have found that the analytic solution and slope field seem to predict very different behavior. Explain. *Hint*: when you verified the analytic solution, what assumptions did you have to make?

(d) Describe the long term behavior of $y(t)$ if $y(0) = 0$. This is a trick question because there is more than one possible answer. Give at least two. This is an example of the general fact that nonlinear equations don't always have a unique solution for each initial condition.

(e) Describe the long term behavior of $y(t)$ if $y(0) = 1/2$. There is only one answer for this condition.

17. An asteroid is detected heading straight towards Earth at 25 km/s. When it is first detected it is 500,000 km from the center of the Earth with negligible velocity. As it falls it experiences a force towards Earth $F = GM_E M_A/r^2$ where M_E and M_A are the masses of the Earth and the asteroid and r is the asteroid's distance from the center of the Earth. You can look up the constants you need online. Be careful to convert all units to SI before entering equations on a computer.

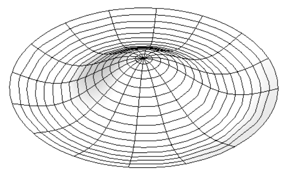
(a) How long will it take to reach the surface of the Earth?

(b) How long would it take an asteroid to reach the surface of Jupiter if were moving straight towards it at 25 km/s starting 500,000 km from the center of Jupiter?

18. A circular drumhead of radius R can have circular standing waves whose amplitude as a function of distance from the center $A(\rho)$ obeys the differential equation

$$A''(\rho) + \frac{1}{\rho}A'(\rho) + k^2 A(\rho) = 0 \tag{0.1}$$

where k is a constant related to the frequency of the wave. The boundary conditions for this equation are that the edges of the drum are clamped down, meaning $A(R) = 0$, and that $A(0)$ must be finite.



(a) Find the general solution to this differential equation. The result should be two special functions, each multiplied by an arbitrary constant.

(b) Using the condition that $A(0)$ must be finite explain why one of the two arbitrary constants in your solution must be zero. Write the resulting solution with one arbitrary constant.

(c) The condition that $A(R) = 0$ does *not* restrict your other arbitrary constant. Instead it restricts the possible values of k. By looking up the values at which the function you found equals zero, find the first three possible values of k for which the condition $A(R) = 0$ can be satisfied for a drum of radius $R = 0.1$ m.

(d) The frequency f is related to k by $f = kv/(2\pi)$, where v is the speed of sound on the drumhead (which depends on its tension). For a drumhead of radius 0.1 m with sound speed $v = 100$ m/s find the first three possible frequencies for circular waves. As a check on your work your answer should come out in units of 1/seconds, otherwise known as Hertz (Hz).

19. *[This problem depends on Problem 18.]* A circular wave on a drumhead is described by the solution you found in Problem 18 multiplied by $\cos(2\pi f t)$, where f is the frequency you found at the end. Using the third value of k you found and the corresponding value of f, make a series of nine plots similar to the one at the beginning of Problem 18, each plot showing the drumhead at a different time. Your final plot should be at the time when it returns to its original shape. (If your program can make animations you can do a single animation instead of the sequence of nine plots.) Use 0.2 for your arbitrary constant (which gives the amplitude of the wave).

20. In quantum mechanics a particle is described by a "wavefunction" ψ that tells you the probabilities of finding the particle in different places. For a particle in a spherical region with no forces acting on it the wavefunction obeys the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} - \left(\frac{l(l+1)}{r^2} - 1\right)\psi(r) = 0$$

where l is an integer related to the particle's angular momentum and the distance from the origin r is expressed in units that allow you to eliminate all other constants from the problem.

(a) Find the general solution for $\psi(r)$.

(b) Using the condition that $\psi(0)$ must be finite, set one of the arbitrary constants in your general solution to zero and write the remaining solution.

(c) The values of r where $\psi(r) = 0$ indicate radii where there is zero chance of finding the particle. Find the first such nonzero radius for the three cases $l = 0$, $l = 1$, and $l = 2$.

21. Superman's enemy Lex Luthor is holding a block of kryptonite, which is deadly to Superman. Superman is attempting to reach Luthor, but the closer he gets to the kryptonite the slower he moves. Assume his velocity is given by $v = -v_s x/(d + x)$ where v_s and d are constants and x is Superman's distance from the kryptonite.

   (a) Sketch the function $v(x)$ and describe what happens to Superman's speed when he is very far from the kryptonite and when he is very close.

   (b) Try to solve this differential equation by hand using separation of variables to find the function $x(t)$. Explain why this *doesn't* work.

   (c) Assume Superman's normal speed when he is far away from kryptonite is 1000 m/s (faster than a speeding bullet) and that his speed drops to half that value when he is 20 m from the kryptonite. Find the values of v_s and d and solve for $x(t)$ numerically assuming he starts 100 m away. If he needs to get within 1 meter of the kryptonite before he can reach it and get rid of it, how long will that take him? (Be careful with sign!)

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

22. **Euler's Method By Hand**

   The simplest technique of numerically solving differential equations is "Euler's method." In this problem, you will use Euler's Method to answer the question: If $dy/dx = y$ and $y(0) = 1$, what is $y(1)$?

   This problem asks us to follow the function from $x = 0$ to $x = 1$. We choose to take this journey in three steps, each of length 1/3.

   (a) You're going to draw a slope field for this equation. To start, draw the axes and draw a small segment of slope at the point $(0, 1)$. Explain how we know that the value of the slope at that point must be 1.

   (b) Our next task is to find the value $y(1/3)$. Using the slope you found, explain how we can know that $y(1/3) \approx 4/3$.

   (c) Calculate the slope of the curve at $(1/3, 4/3)$ and use the result to draw the next entry in your slope field at that point.

   (d) Using the slope you just found, estimate the value of $y(2/3)$.

   (e) Calculate the slope at the new point you just found and use it to draw another entry on your slope field.

   (f) Use the last slope you calculated to estimate $y(1)$.

   (g) We began with the question "What is $y(1)$?" and you answered that question in part (f). Explain why your answer is not exactly correct. Your explanation should enable you to predict whether the actual $y(1)$ is higher or lower than your approximation.

   (h) Solve the equation $dy/dx = y$ with initial condition $y(0) = 1$ analytically and find the exact, correct value for $y(1)$. If you didn't correctly predict whether your approximate answer would be too high or too low rethink your explanation and explain the result you did get.

23. **Euler's Method By Computer**

   [This problem depends on Problem 22.]

   In Problem 22 you used Euler's method to find an approximate value for $y(1)$ given the equation $dy/dx = y$ and the initial condition $y(0) = 1$. You did this in three steps and found a not-so-great approximation to the exact answer.

   (a) Have a computer repeat the calculation, but this time using 10 steps. In other words, starting from the known value $y(0) = 1$ calculate the slope $y'(0)$ and use that to find an approximate value for $y(0.1)$. Then use that to find the slope $y'(0.1)$ and thus the value $y(0.2)$, and so on until you have found a value for $y(1)$. Record the resulting value for $y(1)$. *Hint*: We strongly suggest using a loop to do the ten calculations rather than writing all ten of them out one at a time. This will be faster and easier in this step, and essential for the next one.

(b) Repeat Part (a) with 20 steps instead of 10. You should find that as you increase the number of steps your answer gets closer to the exact answer. *Hint*: If you haven't done so already you should be able to write your calculation in such a way that you can change the number of steps simply by changing one number and rerunning.

(c) Keep doubling the number of steps until your answer for $y(1)$ is within 1% of the exact answer. How many steps do you need and how close is the resulting answer to the exact one?

You've now used Euler's method to get a fairly accurate answer to a problem that you could have answered more easily without it anyway. Of course, the real power of the method is in solving problems you couldn't easily solve analytically! So now consider the equation $dy/dx = tan(x + y)$ with initial condition $y(0) = 1$.

(d) Ask your computer to *analytically* solve this differential equation. What result do you get?

(e) Use Euler's method to find $y(1)$ with four steps. In other words use the slope $y'(0)$ to estimate $y(.25)$ and so on.

(f) Try again with eight steps and keep doubling the number of steps until you get an answer that differs from its predecessor by less than 1%.

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24. Consider the differential equation $dy/dx = y - x^2$.

(a) Have a computer make a slope field for this equation. You may need to do some trial and error to find good ranges for x and y. You should have at least 80 points on your slope field.

(b) Have the computer plot the function $y = 2 + 2x + x^2$ on the same plot as the slope field you just found. You should be able to convince yourself that this function is following the slopes indicated on your slope field, and is thus a solution to the differential equation. (You can also check this analytically.)

(c) Predict how the function will behave if it has an initial condition that places it above the solution you just plotted. How will it behave if it has an initial condition below the one you plotted? Explain your predictions by referring to the behavior of the slope field above and below that solution.

(d) Have the computer numerically solve this differential equation with initial conditions $y(0) = 1$ and $y(0) = 3$. Make a final plot showing the slope field, the particular solution you plotted before, and these two numerical solutions. Does their behavior match your predictions? Explain.