PREFACE

The Four Students: A Math Methods Parable

The professor has just finished a first-semester lecture on definite integrals, possibly unaware that four different types of students are processing the information quite differently. The class comprises *aspiring physicists, aspiring engineers, aspiring mathematicians,* and *bad students.*

- The aspiring physicists are thinking, "Why does subtracting an antiderivative over here from an antiderivative over there give you the area under this curve?"
- The aspiring engineers are thinking, "Of what possible practical value is finding the area under a curve?"
- The aspiring mathematicians are thinking, "How can you claim to have found anything when you haven't rigorously defined 'area under a curve' in the first place?"
- The bad students are thinking, "Just tell me how to do the problems and get the answer you want on the test."

This book is written for aspiring physicists and engineers. For each topic, we hope to clearly answer the questions "Why does this mathematical technique work?" and "How is this used to solve practical problems?" We will fall short of the expectations of true mathematicians, and we hope to continually frustrate the bad students.

Exercises

In almost every section of our book you will find an "Exercise" and a set of "Problems." What's the difference?

The simple answer is that the Problems are, for the most part, independent of each other. You create an assignment that says "Do Section 18.3 Problems 2, 5, 9, 12, and 20." By contrast, an Exercise is an atomic block. You can assign a particular exercise, or you can skip that exercise, but you can't assign "Question 5" from the exercise because it only makes sense in context.

More importantly, the two serve different purposes. Problems are meant to follow a lecture, building and testing the students' skills and understanding of the topics you have discussed in class. Exercises are designed to facilitate active learning.

There are two types of exercises.

Motivating Exercises come at the beginning of each chapter. Their purpose is not to teach math, but to give a practical example of why the student needs the techniques in this chapter.

Discovery Exercises come at the beginning of (almost) every section. Their purpose is to step the student through a mathematical process, such as solving a differential equation or finding a Taylor series. Instead of just being told how to do it, the students do it for themselves.

Some frequently asked questions:

- *Do I need to assign all the exercises?* No. If you are uncomfortable with the process, you may want to try only one or two. We hope you will find them easy to use and valuable, and over time you will use them more, but you will probably never use them all.
- *At home or in class? Alone or in groups?* Mix it up. See what works for you. We sometimes assign them as homework due on the day we are going to cover the material, and sometimes

as an in-class exercise to begin the lecture. You can have students do them individually or in groups, or a mix of the two. One professor we spoke to starts them in class, and then has her students finish them at home—an approach we never even thought of. You will probably keep your students' interest better if you vary your approach.

- *How long do they take?* Some are five minutes or less; some are twenty minutes or even more. Very few of them should take the students more than half an hour.
- *That was all pretty noncommittal. Do you have any solid advice at all?* Actually, we do. First, we hope you will use at least some of the exercises, because we believe they contribute a valuable part of the learning process. Second, exercises should almost always be used *before* you introduce a particular topic—not as a follow-up. You can start your lecture by taking questions and finding out where the students got stuck.

Problems

There are problems at the end of every section, and there are also "additional problems" at the end of every chapter. The "additional problems" give us an opportunity to ask questions where the students don't know exactly what's being covered. (When you look at a partial differential equation in "additional problems" you have to ask yourself: separation of variables? Method of transforms? What's the right approach here?)

We have resisted classifying the problems further, but you will find a general pattern something like this.

- A "walk-through" steps the students carefully through the process we want them to learn. We advise you to generally assign these problems.
- Next often comes a batch of straightforward, unmotivated calculation problems. "Evaluate the following triple integrals in spherical coordinates." They will generally move from easier to harder.
- Then come word problems. Some of these are practical applications; some fill in details that were left out of the explanation; some are just cool ideas that occurred to us while we were brainstorming over Bailey's Irish Cream.
- Finally, in some cases, there are "Explorations." These are harder, more involved, and often longer problems that may stretch beyond the presented material.

Problems without a computer icon (which are most of them) can be done entirely by hand, and should generally require no integration technique beyond u-substitution or integration by parts. A computer icon can mean anything from "This requires an integral that you can do on your calculator" to "This involves heavy use of a computer algebra program such as Mathematica, MATLAB, or Maple." The problems are written in a platform-independent way, and we provide no instruction on any of these computer tools in particular.

Chapter Order and Dependencies

One of the unusual things about Math Methods, as a course, is that it covers a broad variety of loosely (if at all) related topics. It can be taught as a sophomore level course with only two semesters of calculus as a prerequisite (so half the course becomes an introduction to multivariate calculus), or it can be taught as a first-year graduate-level course, or anything in between. It is taught in physics departments, engineering departments, economics departments, and occasionally math departments. All those different courses cover different topics.

So the textbook becomes what Stuart Johnson, our first editor at Wiley, called a "Chinese menu." You look it over and you decide you want this chapter, that chapter, skip three and pick up this one, and so on.

For this reason, we have endeavored as much as possible to make the chapters independent of each other. You don't need our vector calculus chapter to cover our linear algebra chapter, or vice versa. References from one section to another within a chapter are common; references from one chapter to another are rare.

That being said, there are exceptions. Most importantly, pretty much every chapter in the book relies on the information presented in Chapter 1, Introduction to Ordinary Differential Equations. The information is minimal: most of the techniques for *solving* ODEs are deferred to later chapters. But the students have to know what a differential equation is, whether they get it from our chapter or somewhere else. If you are not sure your students come into the course highly comfortable with this material, please start with this chapter before doing anything else. If you want to give it the most minimal treatment possibly, you can get away with doing the three sections "Overview of Differential Equations," "Arbitrary Constants," and "Guess and Check, and Linear Superposition" only. We believe strongly, however, that it is worth your class time to do more.

Beyond that, the first page of every chapter lists prerequisites. You should be able to use that to make sure your students are ready for any given chapter.

Last Word: Communicating Priorities to Students

Here is an experience that took me (Kenny) quite by surprise. I assigned the following problem. "Write the Maclaurin series expansion of e^{-x^2}; then use the first five terms of that series to approximate $\int_0^1 e^{-x^2}\,dx$." Many students came back saying "I had no trouble finding the Maclaurin series, but I didn't understand what you were asking me to do with it."

Just in case you're staring at that sentence with the same dumbstruck look I probably had, I want to stress that these were *not* weak students, and they *did* know how to integrate a polynomial.

And here's my point. We don't just want our students to learn methods; we want them to understand why those methods work, to view those methods in a larger mathematical context, and to be able to apply those methods to physical problems. But students don't develop those skills by being told "You should be able to think for yourself." They develop those skills, just like any others, with practice and feedback. This is particularly relevant for Math Methods, where the skill and the application may be separated by semesters or years. ("What do you mean, quantum mechanics students, you've never heard of a Fourier series? Didn't you take Math Methods?")

Everything in our book is structured to give your students that practice. A discovery exercise says "Don't just listen to me lecture about this; figure it out for yourself." A walk-through says "Let me help you with that important process." The problem after the walk-through, or the later problems in the section, often say "Let's think more deeply about that result" or "Let's see where that came from" or "Let's apply that technique to a circuit." If the explanation stepped through a particularly important derivation that you want your students to understand, there are almost certain to be some problems designed to make sure students followed the derivation. Give your students enough problems to master specific skills—evaluating a line integral, separating variables in a PDE, finding the coefficients in a Fourier series—but assign deeper problems in areas where you want deeper understanding.

Here is a specific example. In the section on Legendre polynomials in the special functions chapter, one problem steps the students through the solution of Hermite's differential equation. If you just want your students to be able to work with Legendre polynomials, you can certainly skip that problem. But if you want your students to follow the derivation of the Legendre polynomials, that problem will force them through every step of the process.

That brings up the more general topic of proofs. It's very rare for our book to prove a theorem before we use it. Much more commonly we present a theorem, show students how to use it, and then step them through the proof in a problem. The explanation will usually point to that problem with language such as "You'll show that this always works in Problem 14." In some cases the problem doesn't prove the theorem, but has the students show that it works for some important cases.

There are two reasons for this unusual structure. First, we believe students follow a proof better after they understand the result that is being proven. Second, we believe very strongly that students follow a proof better if they work through it themselves instead of just reading it.

One of the judgment calls you will have to make, therefore, is which of these proof problems to assign. Our own opinion on this matter, for whatever that's worth, is that the importance of a proof is *not* based on the importance of the result it proves, but on the technique that the proof demonstrates. It's tremendously important for all students to know that the derivative of $\sin x$ is $\cos x$, but very few students can prove it—and that's OK. On the other hand, the proof that $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ in a Fourier series involves an important trick that teaches students what orthogonal functions are and how to find the coefficients of many other such series, so it's worth some investment of class and homework time.

The Most Important Thing We Want to Tell You

Please see http://www.felderbooks.com for all the information you will find in this introduction, exercises formatted for printing, additional problems for every chapter, additional sections including "special applications," answers to odd-numbered problems, and a lot more.

Acknowledgements

Of all the tasks involved in writing a textbook, the one that we most drastically underestimated is writing solutions. We wanted to write the kind of detailed, carefully worked out solutions that we like to give to our students when they need help. Multiply that by 3,000 problems and you have a task that could have held up publication by a year.

So we turned to some of the most talented people we knew—our best current and former students. They stepped up and put in countless hours writing the solutions. (We read and edited all their solutions, so if you see a mistake, it's still our fault.) Along the way they became experts in both LaTeX and the mathematical techniques they were focused on. More importantly for us and for you, they provided the front line of feedback on which problems worked, which problems needed to be revised, and which problems needed to be scrapped. Even if you never look at the solutions, the book you're holding is far better for their dedicated work.

- Introduction to Ordinary Differential Equations: Olga Navros
- Taylor Series and Series Convergence: Szilvia Kiss
- Complex Numbers: Olga Navros, Amanda (Stevie) Bergman
- Partial Derivatives: Jonah Weigand-Whittier
- Integrals in Two or More Dimensions: Mariel Meier
- Linear Algebra I: Szilvia Kiss
- Linear Algebra II: Szilvia Kiss
- Vector Calculus: Meg Crenshaw
- Methods of Solving Ordinary Differential Equations: Alison Grady
- Calculus with Complex Numbers: Alison Grady

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