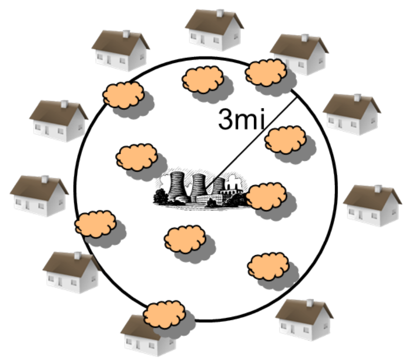
Motivating Exercise: Flowing Fluids

An explosion at the Glogrene nuclear facility has released a cloud of radioactive waste into the surrounding area. The grounds of the facility cover a disk of radius 3 miles, and the waste is uniformly distributed throughout that area with density $\sigma = 1,000,000$ terabecquerels per square mile. (It's a unit used for measuring radioactive materials. Trust us.) Weather satellites have made detailed measurement of wind velocities in the area. They have determined that at any given point (x, y) the wind velocity is roughly $\vec{v} = x^3\hat{i} + (x^2 + y^3)\hat{j}$, where x and y are in miles and $\vec{v}$ is in miles per hour.[1] For instance, at the point $(1, 1)$ the wind velocity is $\hat{i} + 2\hat{j}$. The Glogrene plant is at the origin.



1. What is the wind velocity at the origin?

2. What is the wind velocity at the point $(1, 2)$?

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3. What is the wind velocity at the point $(3, 0)$?

The president has called you in to calculate how quickly radioactive waste is leaving the Glogrene grounds.

4. Draw a diagram of the Glogrene facility. (Just draw a circle with a radius of 3.) Label at the point $(3, 0)$ a small segment of the circumference with height dy.

[1]For simplicity we will assume there is negligible vertical flow, so you can treat this as a two-dimensional problem.

5. Which component of the wind velocity is causing waste to flow out through this segment?

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6. Multiply that component of the velocity times the density of waste, times the height dy, to get the rate at which waste is flowing out through that infinitesimal segment.

To find the total rate at which waste is leaving, you get ready to repeat the above calculation for a differential segment at an arbitrary point on the circle, which will involve finding the component of $\vec{v}$ perpendicular to the segment. Then you will express the amount of waste flowing through that segment per unit time in terms of just one variable (probably the polar angle ϕ) so that you can integrate it to find the total. This is not an approach you want to take if you can avoid it.

As you may have guessed, you can. We assert with no justification that the total rate at which waste is flowing out of the grounds is given by the density of waste times the double integral over the disk of $(\partial v_x/\partial x) + (\partial v_y/\partial y)$ where v_x is the x-component of $\vec{v}$.

7. Calculate $(\partial v_x/\partial x) + (\partial v_y/\partial y)$ for the wind velocity given above.

8. This integral will be easiest in polar coordinates. Rewrite your answer to Part 7 in polar coordinates.

9. Set up and evaluate the double integral of the function you found in Part 7 over the disk. This integral should be straightforward in polar coordinates. Multiply your answer by the density to find the rate at which waste is leaving the Glogrene grounds.

The good news is that you've calculated the rate at which waste is flowing into the community, and using your result the government has created a sensible evacuation plan. The bad news is that you calculated something that we threw at you with no justification. How would this trick apply to other situations? When are you allowed to use it? Why does it work? What is it called?

The last question (which you probably weren't wondering anyway) is easy. We used something called "The Divergence Theorem" to get that formula. For the other answers, you'll have to study vector calculus.