Discovery Exercise for Maclaurin Series - A Polynomial Equivalent of the Sine

An infinitely long polynomial is referred to as a "power series":

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Our goal in this exercise is to write a power series that equals $\sin x$ for all values of x: that is,

$$\sin x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \quad (1)$$

In writing Equation 1, we have not actually solved anything. Equation 1 is merely the assertion that there exists a power series expansion of the function $\sin x$, if we can only find the right coefficients. In finding these coefficients for the sine function, we will develop a general process for finding the power series coefficients for any given function for which such a power series exists.

We start with a simple observation: if the equation is true for all x-values, then it must certainly be true for $x = 0$.

1. Plug the value $x = 0$ into both sides of Equation 1. Note that all the terms on the right except the first one vanish, enabling you to solve for the first coefficient, c_0. (As one student wryly observed, "One down, infinity to go.")

2. Now, take the derivative of both sides of Equation 1. (If two functions are equal everywhere, their derivatives must also be equal.)

3. Plug the value $x = 0$ into both sides of the equation you wrote down in the previous step. Once again, almost all the terms on the right disappear, enabling you in this case to solve for the second coefficient, c_1.

4. Repeat steps 2 and 3: take the derivative of both sides, and then plug $x = 0$ into both sides, until you have found all the coefficients through c_8.

If you've done everything correctly so far your first few terms should be:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

and you should have found at least one more term after that. The first term $f(x) = x$ is the linear approximation to $\sin(x)$ at $x = 0$. When you include the first two terms, $f(x) = x - x^3/(3!)$ should be a *better* approximation of $\sin(x)$ near $x = 0$. And $f(x) = x - x^3/(3!) + x^5/(5!)$ should be better still, and so on. For the function $\sin x$, and for many (though not all!) functions, we can continue adding terms to estimate the function as accurately as we need.

We refer to these successive approximations as "partial sums" of the series, and write:

First partial sum:	S_1	$=$	x
Second partial sum:	S_2	$=$	$x - x^3/(3!)$
Third partial sum:	S_3	$=$	$x - x^3/(3!) + x^5/(5!)$

Let's see how these partial sums work as approximations for the sine function.

5. Fill in all the numbers in the following table to at least five decimal places accuracy. These values of x are in radians.

x	$\sin x$	$S_1(x)$	$S_2(x)$	$S_3(x)$	$S_4(x)$
1					
2					
3					

6. Answer in words: looking at the *first row* in your table (for $x = 1$), what do we learn about successive sums of this series?

7. Answer in words: looking at *all three rows* of your table, what do we learn about this series as x gets farther from zero?

8. Now, use graphing software, a spreadsheet, or a graphing calculator to make *graphs* of all five functions: $\sin x$, $S_1(x)$, $S_2(x)$, $S_3(x)$, and $S_4(x)$, on the interval $-\pi \leq x \leq \pi$.

9. Answer in words: what do the five graphs tell us about successive sums of this series?