Discovery Exercise for Geometric Series

Consider the sum: $5 + 15 + 45 + 135 + 405$. This is called a "geometric series," meaning each term is obtained by multiplying the previous term by a constant. The constant is known as the "common ratio," or r. In this example, r is 3.

You could add up those five terms in a minute, by hand or with a calculator. You could also presumably add up the first six terms of that series, since you can readily figure out that the next term is $405 \times 3 = 1215$. But what if we asked you to add up the first twenty terms, or the first hundred?

Amazingly, there is a trick—we call it the "geometric series trick"—that quickly adds up all the terms of any geometric series. You're going to use it to add up just the five-term series we gave above, but you'll see that it would be equally quick for a series of any length. Later we'll show you how to use it to add up an infinite series.

You begin by assigning the letter S to the sum you are looking for:

$$S = 5 + 15 + 45 + 135 + 405$$

1. What is $3S$? (Write it out term by term; don't add them up.)

2. Now, write the equation $3S =$(what you just said), and below that, write our original equation for S. Then subtract the two equations. You will find that, on the right side of the equal sign, all but two of the terms cancel out.

$$\begin{aligned} 3S &= \\ S &= \end{aligned}$$

3. Solve the resulting equation for S.

   See Check Yourself #9 at felderbooks.com/checkyourself

4. Repeat this process with the geometric series $1000+100+10+1+1/10+1/100$. Note that in this case $r = 1/10$, so you will have to begin by writing a formula for S and one for $(1/10)S$. Once again, make sure you get the correct answer (which is 1111.11 in this case).

5. Now let's generalize this process. The general geometric series is...

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \ldots a_1 r^{n-1}$$

   Multiply this whole equation (both sides) by r. Then, subtract that new equation from the original equation. You should once again find that all the terms on the right but two cancel, leaving a formula you can solve for S. Do that. When you are done, you should have a general formula for the sum of any geometric series that starts at a_1 and goes on for n terms with common ratio r.