Computer Problems for Taylor Series and Series Convergence

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

1. The drawing below depicts a familiar situation: a mass m swings at the end of a massless rope of length L, subject to the forces of gravity and the tension of the string. It can be shown that this pendulum will obey the

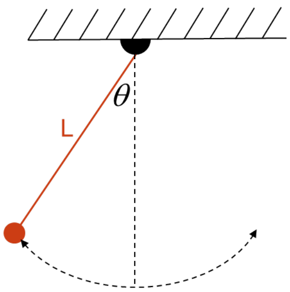

Figure 1: Simple pendulum

differential equation $d^2\theta/dt^2 + (g/L)\sin\theta = 0$. (You can work that out yourself if you've taken an introductory mechanics class.) Unfortunately, this equation is not easy to solve!

   (a) Find the function that serves as a linear approximation to $\sin(\theta)$ around $\theta = 0$.
   (b) Replace the function $\sin\theta$ with its linear approximation.
   (c) Solve the resulting differential equation.
   (d) Describe in words the motion resulting from your solution.
   (e) For what kinds of θ-values is your solution valid?

2.  *[This problem depends on Problem 1.]* A grandfather clock uses a pendulum similar to the one described in the previous problem to keep time. The pendulum swings in a narrow box so that it always has a small angle θ. The box is designed this way so that the linear approximation will be accurate. In this problem you will estimate how small those oscillations need to be. The parameter g is 9.8 m/s^2 and you can take $L = 1m$.

   (a) Using your solution from the previous problem, find the period of the pendulum for small oscillations. You should find that your answer is independent of the amplitude, as long as it is small enough that your linear approximation can be considered valid.
   (b) Use a computer to solve the original (not approximate) equation $d^2\theta/dt^2 + (g/L)\sin\theta = 0$ with initial conditions $d\theta/dt = 0$, $\theta = 5°$ and plot the result. (*Caution: Many computer programs expect angles to be in radians, so you may have to convert before entering θ.*) Use trial and error to find a final time that allows you to see at least two full oscillations.
   (c) Have the computer find the period of the solution you found in Part (b). You should be able to do this by asking it to find two successive roots of the solution and subtracting them.
   (d) Repeat Parts (b) and (c) for initial angles between 10° and 45° in steps of 5°. Make a plot of oscillation period vs. amplitude in the range $5° \leq A \leq 45°$. Be sure to choose a scale for the vertical axis that lets you see how the period changes with amplitude.
   (e) Describe in words what happens to the period as the amplitude decreases. Does it keep increasing, keep decreasing, level off at some value, or do something else entirely?
   (f) Using your results from Parts (a) and (d) estimate the maximum amplitude that a grandfather clock could swing at if you wanted its period of oscillation to be within 1% of the one given by the linear approximation.

3. Consider the equation $dy/dt = (1 + \sin y)^{-2}$.

   (a) Replace the right hand side with its linear approximation around $y = 0$ and have a computer solve the resulting differential equation *analytically* with initial condition $y(0) = 0$.

   (b) Numerically solve the original equation with the same initial condition and plot this together with your approximate solution out to time $t = 5$. You should find that the two solutions do *not* match well at late times.

   (c) Explain why the linear approximation that you used was not a good approximation for this problem. (*Hint: Look at how y behaves at late times.*) Don't just say that the two solutions in Part (b) looked different; explain why the linear approximation, which works so well for many problems, was not very accurate for this one.

4. Consider the equation $dy/dt = 2 - e^y$.

   (a) Replace the right hand side with its linear approximation around $y = 0$ and have a computer solve the resulting differential equation *analytically* with initial condition $y(0) = 0$.

   (b) Numerically solve the original equation with the same initial condition and plot this together with your approximate solution out to time $t = 5$. You should find that the two solutions do *not* match well at late times.

   The asymptotic value of y at late times (also known as the "steady state" value) is the value of y such that $dy/dt = 0$.

   (c) Find the steady state value of y.

   (d) Find a linear approximation to the right hand side of the differential equation about the steady-state y-value.

   (e) Now have the computer analytically solve the differential equation using this new linear approximation (still with initial condition $y(0) = 0$).

   (f) Plot the three solutions together: the numerical solution to the original equation and the analytical solutions using the two different linear approximations.

   (g) In which region of the plot does the linear approximation from Part (a) approximate the full solution better? In which region of the plot does the linear approximation from Part (d) do better? Why? Under what circumstances would each of these be a more useful approximation to use for a physical problem?

5. Consider the differential equation $d^2f/dt^2 = 2\left(e^{-3f} - 1\right)$.

   (a) Replace the right hand side with its linear approximation around $f = 0$ and solve the resulting differential equation. (You can do this by hand or with a computer.) Solve for the arbitrary constants with initial conditions $f(0) = 0$, $f'(0) = 0.2$.

   (b) Numerically solve the original differential equation (without the approximation) using those same initial conditions. Plot the approximate and exact (numerical) solutions on the same plot, clearly marking which is which. Use trial and error to find a final time that allows you to see at least two full oscillations.

   (c) Repeat Parts (a) and (b) for initial conditions $f(0) = 0$, $f'(0) = 2$.

   (d) Describe in words the differences between your plots in Parts (b) and (c). In which case was the linear approximation more accurate? Why?

In Problems 6–8 make a plot showing the given function and the first three partial sums of its Maclaurin series in the range $-5 \le x \le 5$. In calculating the partial sums you should skip terms that equal zero, so for example the second partial sum of $1 - x^2/2 + x^4/24$ would be $1 - x^2/2$. Make sure it's clear which curve is the function and which ones are the partial sums. Estimate visually the value of x where each partial sum stops being a good approximation. (There is no precise correct answer for this question; just make a reasonable estimate based on looking at your plot.)

6. $f(x) = \cos x$

7. $f(x) = 1/(2 + \cos x)$

8. $f(x) = \sin\left(e^{-x^2}\right)$

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9. In this problem you'll use the function $f(x) = 0.5\sin(x^2) + 1.5\sin(x) + 0.2$ to visually demonstrate the convergence of a Maclaurin series to the function it is approximating.
   (a) Plot $f(x)$ in the range $-5 \le x \le 5$. Describe its shape in 20 words or less.
   (b) Display on one plot $f(x)$ and every tenth partial sum up through the hundredth. In other words, display the zeroth order sum, the tenth order, the twentieth, and so on up through the hundredth—eleven partial sums in all. Plot $f(x)$ as a thick, black curve and the partial sums as thin lines, each of a different color. The horizontal axes should run from -5 to 5 and the vertical axes from -3 to 3.
   (c) Describe how the plot changes as you go from one partial sum to the next.
   (d) Visually estimate the values of x (one low and one high) where the 100th order partial sum stops being a good approximation to $f(x)$. How many local maxima and minima does this partial sum trace out nearly identically with $f(x)$?

10. **Calculating π.** If you want to know the value of π to several decimal places you can of course just plug it into your calculator. But how were those decimals found, and how do computers nowadays calculate π to *billions* of decimal places? Many of the methods used involve Taylor series.
    (a) Find the Maclaurin series for $\tan^{-1} x$ out to seventh order. (You can get the derivatives you need from a computer program, a table of derivatives, or just by searching the Web.) From there you should be able to spot the pattern and be able to write it down to any order you want. Use this to write an equation of the form $\tan^{-1} x = \sum_{n=0}^{\infty}$ <something>.
    (b) Plug $x = 1$ into the equation you just wrote. This should give you an expression involving π on the left and a series involving simple fractions on the right. You can thus use this series to calculate π to any desired accuracy. Use the seventh order approximation to find an estimate of π.
    (c) Now have a computer calculate and store the partial sums of this series from $n = 1$ to $n = 1000$. Generate a plot of the partial sum vs. n. The plot should converge towards the value of π.

    (In practice this series is not a great way to calculate digits of π since it converges so slowly, as you saw in your plot. There are other, more complicated series that converge more quickly.)

11. (a) Find the Taylor series for $\sin x$ around $x = \pi$. Express your answer in summation notation.
    (b) Use the first three non-zero terms of your series to estimate $\sin(3)$.
    (c) Calculate the percent error in your estimate from Part (b).
    (d) Draw graphs of $\sin x$, and of the first three partial sums of your series, for $\pi/2 \le x \le 5\pi/2$. Does each partial sum approximate the original function better than the previous?

12. (a) Find the Taylor series for $\cos x$ around $x = \pi/3$. (*Hint:* $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.)
   (b) Use the first three non-zero terms of your series to estimate $\cos(1)$.
   (c) Calculate the percent error in your estimate from Part (b).
   (d) Draw graphs of $\cos x$, and of the first three partial sums of your series, for $-\pi/2 \leq x \leq \pi$. Does each partial sum approximate the original function better than the previous?

13. **Chaos Theory**

   Biologists model constrained population growth by the recursive sequence $p_{n+1} = kp_n(1 - p_n)$, where p_n is the population after n generations and k is a positive constant called the "growth rate." p_n is always between 0 (no population) and 1 (full capacity): $p = 1/2$ does not mean the population is half an animal, but that the population is at half its capacity.

   (a) Letting $k = 1.2$ and $p_1 = 0.1$, use a computer program to rapidly generate at least thirty generations. You will see the population approaching an equilibrium point. What is it?
   (b) With k still set to 1.2, experiment with different values of p_1 ranging from 0.1 to 0.9. Note that the resulting numbers differ each time but they approach the same limiting value.
   (c) Repeat the experiment with k set to 1.5, 2.0, 2.3, and 2.7. Record in a table the k-values and the resulting equilibrium points. For each value of k be sure to try at least two values of p_1.
   (d) Repeat the experiment with $k = 3.1$. What happened?
   (e) Repeat the experiment with $k = 3.2$. The results are similar, but not identical, to the results in Part (d). Describe them.
   (f) Repeat the experiment with $k = 3.5$. What happened now?
   (g) Finally, repeat the experiment with $k = 3.7$.

   With $k = 3.7$ the sequence represents what mathematicians call "chaos." There is no discernable pattern to the numbers. If you repeat the experiment with the exact same starting value, of course you will get the same results. However, if you change the starting value, even by a very small amount, the resulting numbers will be completely different. (This property goes by the imposing name "sensitive dependence on initial conditions.")

   (h) Chaotic behavior—including sensitive dependence on initial conditions—has been shown to apply to real-world systems including the weather and the stock market. What implications does this have for scientists attempting to predict the behavior of such systems?

14. The equation below gives the asymptotic series expansion for erfc x.

$$\text{erfc } x \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!(2x)^{2n}}$$

   Suppose you wish to use this series to approximate erfc 3.

   (a) Evaluate terms of this series expansion at $x = 3$. You should see that the terms decrease in magnitude for a while and then increase. What term has the smallest magnitude?
   (b) In Part (a) you found the n-value that minimizes the magnitude of the terms in this series. Make a graph of the *partial sums* of this series as n goes from 0 up to twice that value. Include on your graph a label showing the correct value of erfc 3. Describe the behavior you see.
   (c) How many terms of this series do you need to obtain an approximation accurate to within 0.1%?
   (d) Repeat Parts (a)–(c) for erfc 5. How is the behavior the same, and how is it different?

15. The "exponential integral" is defined as Ei $x = -\int_{-x}^{\infty}(e^{-t}/t)dt$. The following series converges to Ei x for all $x \neq 0$: Ei $x = \gamma + \ln|x| + \sum_{n=1}^{\infty} x^n/(nn!)$. Here γ is the "Euler-Mascheroni constant," roughly equal to 0.5772. You can also represent Ei x with the following asymptotic expansion, valid in the limit $x \to -\infty$.

$$\text{Ei } x \sim \frac{e^x}{x}\sum_{n=0}^{\infty}\frac{n!}{x^n}$$

You'll use both of these series to approximate Ei(-10).

(a) Prove that this asymptotic series diverges for any value of x

(b) Calculate the value of Ei(-10) to at least 5 decimal places.

(c) Calculate the 40^{th} partial sum of both series. Is either one a good approximation?

(d) Calculate the 30^{th} partial sum of both series. Is either one a good approximation?

(e) Calculate the 3^{rd} partial sum of both series. Is either one a good approximation?

(f) Plot the partial sums of both series up to $N = 50$ as a function of N (the maximum value of n used in the partial sum). Show on your plots the correct value of Ei(-10). Describe how each series behaves.

(g) Why is it useful to have a divergent, asymptotic expansion even though there is a convergent series that works for this function?

16. The "error function" is defined by the following integral.

$$\text{erf } x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

Its asymptotic expansion is

$$\text{erf } x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!(2x)^{2n}}$$

(a) Write the Maclaurin series for e^t. (You can derive it or look it up.) From there you should be able to easily generate the Maclaurin series for e^{-t^2} and from there the error function. You should not need a computer for this part, although you can use one to check your answer when you're done.

You now have two different series that can be used to estimate erf x. In the remainder of this problem you will compare these two series using a computer.

(b) Use the $n = 3$ partial sum of each series to estimate erf $1/2$. Which estimate is more accurate?

(c) Use the $n = 3$ partial sum of each series to estimate erf 30. Which estimate is more accurate?

(d) On one plot, show erf x and the $n = 3$ partial sum of both series for $0 \leq x \leq 30$. Choose a vertical range that allows you to see when each series is and isn't a good approximation to the function, and estimate the ranges in which each one gives a good estimate.

(e) One one plot, show the partial sums of both series with $x = 5$ as a function of N, the highest n value of the partial sums. Estimate the range of partial sums for which each series gives a good approximation to erf 5.

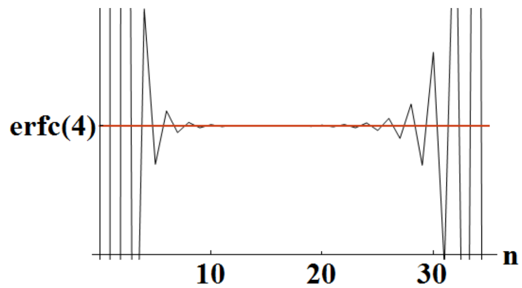

Figure 2: Partial sums of the asymptotic series for erfc x, evaluated at $x = 4$

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

17. The equation below gives the first term in the series expansion of erfc x.

$$\text{erfc } x = \frac{e^{-x^2}}{\sqrt{\pi}\, x} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^2} e^{-t^2} dt$$

   (a) Use integration by parts on that series to find the next term. Your final answer will be in the form erfc x=<*two terms plus an integral*>. *Hint:* You cannot integrate e^{-t^2} but you can integrate te^{-t^2}.

   (b) After n integrations the series looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$. Assume the remainder term is given by the following integral with some coefficient c_n.

$$R_n = c_n \int_x^\infty \frac{1}{t^{2n}} e^{-t^2} dt$$

   Perform the next integration by parts to find the term a_{n+1} and the remainder R_{n+1}. You should verify that the pattern continues because R_{n+1} looks like R_n with n replaced by $n+1$ and a different coefficient.

   (c) Do one more integration by parts to find a_{n+2}. (Don't worry about the remainder integral this time.) Simplify the ratio $|a_{n+2}/a_{n+1}|$.

   (d) Given a fixed value of x, for which values of n will a_{n+2}/a_{n+1} be greater than 1, and for which will it be less than 1?

   (e) Using your answer to Part (d), explain why Figure 2 looks like it does. In particular, explain how you can use that answer to predict the value on the horizontal axis at which the terms switch from converging towards a finite value to diverging away from it.

   Your results in this problem demonstrate that for any fixed x the asymptotic series for erfc x should converge towards a finite value for a while, and then start diverging above a certain value of n. Problem 18 will continue this argument by arguing that the finite value the series converges to is in fact erfc x.

18. **The Remainder of the Asymptotic Expansion for erfc** *[This problem depends on Problem 17.]*

   At any particular value of n, the asymptotic series for erfc x looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$, where R_n is the exact difference between the series expansion and the correct value of erfc x. In this problem you're going to show that as n increases this remainder term decreases for a while, indicating that the series is getting closer to erfc x, but that it starts to increase past a certain value of n. In fact you'll show that this is approximately the same value of n at which the terms a_n go from decreasing to increasing.

   In Problem 17 you showed that if $R_n = c_n \int_x^\infty t^{-2n} e^{-t^2} dt$ then $R_{n+1} = -c_n[(2n+1)/2] \int_x^\infty t^{-(2n+2)} e^{-t^2} dt$. The coefficient in front of the R_{n+1} integral is bigger than the one in front of the R_n integral by a factor of $(2n+1)/2$. At the same time, however, the integrand decreased by a factor of t^2, and it's harder to say what effect that has on the entire integral. The key to figuring that out is to notice that the decaying exponential causes the integral to be dominated by values of t very close to x.

(a) Numerically calculate $\int_x^\infty t^{-2}e^{-t^2}dt$ and $\int_{x+.1}^\infty t^{-2}e^{-t^2}dt$ for values of x ranging from 2 to 20. Show that for sufficiently large x over 90% of the entire integral comes from $x < t < x + .1$. Estimate the lowest value of x for which is true.

(b) Numerically calculate the ratio of $\int_x^\infty t^{-2}e^{-t^2}dt$ to $\int_x^\infty t^{-4}e^{-t^2}dt$ for values of x ranging from 2 to 20. On the same plot, plot x^2.

(c) What does dividing the integrand by t^2 do to the value of the integral? Explain how your answer follows from the plots you made.

(d) Putting together your answers so far, write a simple approximation for R_{n+1}/R_n, valid for large x.

(e) Using your answer to Part (d), estimate the value of n at which the remainder term switches from decreasing to increasing as you increase n. Check your answer by verifying that it correctly predicts the appearance of Figure 2.

(f) Explain how the results you've derived in this problem lead to the two properties that we said define an asymptotic series. *Hint:* Remember that this integral represents the difference between the original function and the n^{th} term of the asymptotic series!

For Problems 19–22 use a computer to calculate partial sums of the Taylor series for the function about the midpoint of the domain. On one plot, show the function in black and its partial sums in different colors. You should show enough partial sums to clearly see how they are changing as you add more terms, and your final partial sum should match the function well throughout the domain.

19. $\sin x$, $-5 \le x \le 5$

20. e^{-x^2}, $-2 \le x \le 2$

21. e^{-x^2}, $0 \le x \le 2$

22. $\sin^3 x$, $-\pi/2 \le x \le \pi$

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23. Let $f(x) = e^x + e^{x^2}$. Use a tenth order Maclaurin series to estimate $f(.8)$. Use the Lagrange remainder to place an upper bound on the error of this approximation, and verify that the upper bound is correct.

24. The differential equation $dx/dt = x + \sin x + \cos x$ is nonlinear and has no simple solution.

    (a) Use a Maclaurin series to approximate the right side of this equation with a linear function of x, valid when $x \approx 0$.

    (b) Solve this approximate differential equation with the initial condition $x(0) = 0$.

    (c) Do you expect your approximation to remain valid at late times? Explain, using the solution you found.

    (d) Plot the approximate solution you found and the numerical solution to the original differential equation, from $t = 0$ to $t = 1$. Does the behavior match your prediction? Explain.

25. Like π, e is an irrational number that has been calculated to large numbers of digits. (As of late 2010 the first trillion digits were known.) For this problem we'll let you get away with doing the first 10000. Use the Maclaurin series for e^x with $x = 1$ to find successively better approximations of e. Keep adding terms up to and including the first term that is smaller than 10^{-10000}. Give as your final answer the 9997^{th} through 10000^{th} digits. (Count the initial 2 as the first digit.)