9.9 Additional Problems

If a problem is not marked with a computer icon then you should be able to do all integrals by hand and/or by using the table in Appendix G.

9.128 In this problem you are going to create a Fourier series for the function $f(x) = 3x$ on the domain $[0, 2]$.

- **(a)** The process begins with a periodic extension of $f(x)$ (neither even nor odd). What is the period of this extension? What is the value of L in the Fourier series formulas?
- **(b)** Write the formulas for a_0, a_n and b_n without yet evaluating the integrals.
- **(c)** Plot the periodic extension from $x = -4$ to $x = 4$.
- **(d)** When you evaluate your integrals you can choose to use any limits of integration that are separated by one full period. Which full period is easiest to use? Explain why by referring to your plot.
- **(e)** Finish finding the Fourier series for $f(x)$.

For Problems 9.129–9.133, find the Fourier series of the indicated function twice, first using sines and cosines and then using complex exponentials. If it is given over a finite domain assume that it extends periodically beyond that domain. (Do not assume an even or odd extension.)

9.129 $f(x) = \sin(2x) + \cos(3x)$

9.130 $f(x) = x + 3$, $-\pi < x < \pi$

9.131 $f(x) = \sin^5 x$. *Hint*: don't just naively accept what the computer gives you if it doesn't make sense.

9.132 $f(x) = x^2 \sin x$, $0 < x < \pi$

9.133 $f(x) = \sin(2x) + \cos(\pi x)$, $0 < x < 1$. *Hint*: the easiest way to evaluate the sine and cosine integrals is to rewrite them as complex exponentials.

9.134 In each part of this problem find a sine/cosine Fourier series for the given function on the given domain. In Part (a) you will do this in the usual way, by evaluating integrals for the coefficients. Using that Fourier series, you should be able to do the remaining parts without evaluating any integrals.

- **(a)** $f(x) = x$ from $x = 0$ to $x = 1$
- **(b)** $g(x) = x + 1$ from $x = 0$ to $x = 1$
- **(c)** $h(x) = -x$ from $x = 0$ to $x = 1$
- **(d)** $i(x) = 2x$ from $x = 0$ to $x = 1/2$

9.135

- **(a)** Write a Fourier series that represents the function $y = x^2$ from $x = -20$ to $x = 20$.
- **(b)** Draw the 1st, 5th, 20th, and 100th partial sums of the resulting series on one plot, along with $y = x^2$.

9.136 The function $f(x)$ equals $2x$ on $-3 \le x \le 3$ and then repeats forever with a period of 6.

- **(a)** Draw $f(x)$ showing at least three full periods.
- **(b)** Create a Fourier series for $f(x)$ based on sines and cosines.
- **(c)** Create a Fourier series for $f(x)$ based on complex exponentials.
- **(d)** Is $f(x)$ odd, even, or neither? How is this reflected in both Fourier series?

9.137 The function $g(x)$ equals $6 - 2x$ on $0 \le x \le 3$ and then repeats forever with a period of 3.

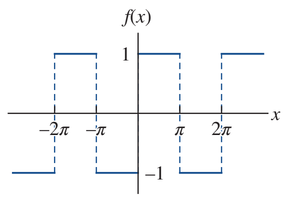
- **(a)** Draw $g(x)$ showing at least three full periods.
- **(b)** Create a Fourier series for $g(x)$ based on sines and cosines.
- **(c)** The function $g(x)$ is neither odd nor even, but $g(x) - 3$ is. Is it even or odd, and how can you see that in the Fourier series you just wrote?
- **(d)** Create a Fourier series for $g(x)$ based on complex exponentials.

9.138

- **(a)** Show that the functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ for all positive integers n form an orthogonal set on the interval $[-L, L]$. (If you're stuck you may find it helpful to look at Section 9.3 Problem 9.32.)
- **(b)** Show that if you add any function $\sin(k\pi x/L)$ with non-integer k the set is no longer orthogonal.

9.139 The drawing below, repeating the same pattern forever to both left and right, represents the function $y(x, t)$ during the time $-2\pi < t < 2\pi$. Then the function suddenly switches: all the

$y = 1$ values move down to $y = -1$ and vice versa, and it stays like that for 4π seconds. Then it switches back…and so on forever.



(a) What is the period of this function in x?

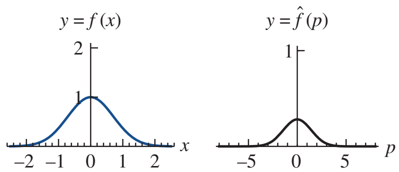
(b) What is the period of this function in t?

(c) Is $f(x, t)$ even in x, odd in x, or neither?

(d) Is $f(x, t)$ even in t, odd in t, or neither?

(e) Create a multivariate Fourier series representation of $y(x, t)$.

9.140 The drawings below represent $y = e^{-x^2}$ and its Fourier transform.



(a) Copy these drawings onto two separate graphs and label them $y = f(x)$ and $y = \hat{f}(p)$.

(b) Add to the first drawing a different graph labeled $g(x) = 2f(x)$, and add to the second drawing a different graph labeled $y = \hat{g}(p)$. These drawings do not need to be exact, but they should show the correct transformations of the original graphs. No computation should be required.

(c) Copy the original drawings again onto separate graphs. Then add new graphs labeled $h(x) = f(2x)$ and $y = \hat{h}(p)$.

9.141 Let $f(x) = x^{-1/3}$ from $-\pi$ to π and repeat thereafter.

(a) This function only satisfies the Dirichlet conditions if $\int_{-\pi}^{\pi} f(x)dx$ is finite. Show that it is. (Because of the vertical asymptote, you have to break it up into $\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$ and show that both parts are finite.)

(b) Use a computer to generate the 20th partial sum of the Fourier series for this function between $x = -\pi$ and $x = \pi$, and then plot that series from $x = -2\pi$ to $x = 2\pi$.

(c) The resulting graph should look a lot like $x^{-1/3}$ between $-\pi$ and π. What does the graph do at $x = 0$ and why? What does the graph do at $x = \pi$ and why?

(d) What are the values of the Fourier series at $x = 0$ and $x = \pi$? Why do these values make sense?

9.142 A simple harmonic oscillator with an external driving force obeys the following differential equation.

$$\frac{d^2x}{dt^2} + 4x = f(t) \qquad (9.9.1)$$

(a) Find the complementary solution that you get if you replace $f(t)$ with 0. Your answer should have two arbitrary constants in it.

(b) Let $f(t) = \sin t$. Find a particular solution that satisfies the differential equation with no arbitrary constants and add it to the complementary solution to get the full solution. *Hint*: guess a solution of the form $x(t) = k \sin t$ and plug it in to find out what k has to be.

(c) Find the general solution if $f(t) = \sin(\omega t)$ where ω is an unspecified constant.

(d) Find the general solution if $f(t)$ is a square wave, equal to 1 from $t = 0$ to $t = 1$, -1 from $t = 1$ to $t = 2$, and repeated periodically thereafter. *Hint*: At the risk of stating the obvious, the solution will involve taking a Fourier series.

9.143 *[This problem depends on Problem 9.142.]* Find the general solution to Equation 9.9.1 if $f(t)$ is a square wave equal to 2 from $t = 0$ to $t = 1$, 0 from $t = 1$ to $t = 2$, and repeated periodically thereafter. *Hint*: you can use your work from Problem 9.142 but you will need to add an additional particular solution.

9.144 The temperature distribution on a disk is best described in polar coordinates, where ρ goes from 0 (the center) to the radius R (the rim), and ϕ goes from 0 to 2π. If the disk has no heat sources or sinks then the temperature distribution throughout the disk is determined by the “boundary condition”: the temperature along the rim. In Chapter 11 you will show (here you can take our word for it) that if the outer edge is held at $T(R, \phi) = T_0 \sin(k\phi)$

then the temperature throughout the disk will obey the following equation.

$$T(\rho, \phi) = T_0 \left(\frac{\rho}{R}\right)^k \sin(k\phi)$$

(a) What are the units of the constants k and T_0?

(b) If the temperature on the edge of the disk is continuous, what values of k are allowed?

(c) If you start at a point on the edge where the temperature is T_0 and move directly toward the center, describe how the temperature will change as you move. What if you start at a point on the edge where the temperature is $-T_0$? What about zero?

(d) Write the temperature distribution on the disk if the temperature on the outer ring is $10\sin(4\phi)$.

(e) Write the temperature distribution on the disk if the temperature on the outer ring is $10\sin(5\phi)$.

(f) Describe qualitatively the differences in the two temperature distributions you just wrote down. (You should note two important differences.)

If the temperature on the outer ring is a sum of sines then the temperature throughout the disk will be the sum of the corresponding solutions. For example, if $T(R,\phi) = \sin(\phi) + 15\sin(2\phi)$, then $T(\rho,\phi) = (\rho/R)\sin(\phi) + 15(\rho/R)^2\sin(2\phi)$.

(g) Write the temperature distribution on the disk if the temperature on the outer ring is $2\sin(3\phi) - 6\sin(4\phi)$.

(h) Now consider a disk whose outer ring is held at $T = T_0$ for $0 < \phi < \pi$, and at $T = -T_0$ for $\pi < \phi < 2\pi$. Rewrite the outer ring temperature $T(R,\phi)$ as a Fourier series. Then write the temperature distribution for the entire disk $T(\rho,\phi)$ as a Fourier series.

9.145 **The Gibbs Phenomenon** If a function $f(x)$ satisfies the Dirichlet conditions then its Fourier series converges to its value where it is continuous and converges to the average of its left and right limits at jump discontinuities. However, the convergence at such discontinuities suffers from a problem known as the Gibbs phenomenon. You'll explore this by considering the function $f(x)$ defined as the odd extension of $f(x) = 1$ from $x = 0$ to $x = 1$.

(a) Find the Fourier sine series of $f(x)$.

(b) Plot the partial sum of the Fourier series including the first three non-zero terms. Your plot should go from $x = -1$ to $x = 1$ and should include horizontal lines at $y = -1$ and $y = 1$ for reference.

(c) For negative x the series should be near -1 and for positive x its should be near 1, but in going from negative to positive it "overshoots" by a bit, going higher than 1. Looking at your plot, estimate the amount it overshoots by.

(d) Show all of the partial sums up through the sixth non-zero term together on one plot. As you use more terms, you should find that the amount by which the series overshoots 1 does *not* decrease, but that it moves back down near 1 more quickly.

(e) Plot the 50^{th} partial sum of the Fourier series and show that it still overshoots by the same amount. Zoom in your plot enough to estimate the value of x at which it returns back to 1 after overshooting.

(f) The fact that a Fourier series overshoots a jump discontinuity by an amount that doesn't decrease as you add more terms is the Gibbs phenomenon. Explain how we can still say that the series converges to the function even though this occurs.