CHAPTER 8

Vector Calculus (Online)

8.12 Additional Problems

In Problems 8.190–8.195 find all the field derivatives (gradient, divergence, curl, Laplacian) that can be calculated for the given fields, some of which are scalars and some of which are vectors.

8.190 $f = x^2 - y^2$

8.191 $\vec{f} = 3x^2\hat{i} - 3x^2\hat{j} + (y-z)^2\hat{k}$

8.192 $f = (\rho/z)\cos\phi$

8.193 $\vec{f} = \rho\sin\phi\,\hat{\rho} + \rho\cos\phi\hat{\phi} + \left(\rho^2/z\right)\,\hat{k}$

8.194 $f = r^2\sin\theta\cos\phi$

8.195 $\vec{f} = r\sin\theta\hat{r} + r\cos\phi\hat{\phi}$

In Problems 8.196–8.202 find a function F such that $\vec{\nabla}F = \vec{v}$ or prove that none exists. (This is one way of determining if $\vec{v}$ is conservative. If it is, your F is *negative* the potential function for $\vec{v}$.)

8.196 $\vec{v} = 3\hat{i} + 2\hat{k}$

8.197 $\vec{v} = xy\,\hat{i} + xy\,\hat{j}$

8.198 $\vec{v} = 3y\hat{i} + 3x\hat{j} + 3z\hat{k}$

8.199 $\vec{v} = e^x\left[\sin(x+2y) + \cos(x+2y)\right]\hat{i} + 2e^x\sin(x+2y)\hat{j}$

8.200 $x(\sin y)e^z\hat{i} + \left(y^2 - x(\cos y)e^z\right)\hat{j} + \left(2z + x(\sin y)e^z\right)\hat{k}$

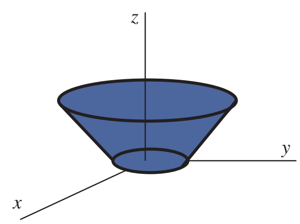
8.201 $\vec{v} = ze^x\hat{i} + \hat{j} + e^x\hat{k}$

8.202 $\vec{v} = \left(ye^{x+z} - 2(x+y+z)\right)\hat{i} + \left(e^{x+z} - 2(x+y+z)\right)\hat{j} + \left(e^{x+z} - (x+y+z)\right)\hat{k}$

Problems 8.203–8.207 ask questions that make no mention of the divergence theorem or Stokes' theorem. But you can use these theorems to make many of these questions easier.

8.203 The electric field in a region of space is $\vec{E} = kx\hat{i}$, where k is a constant. Find the flux of this field through the spherical surface $x^2 + y^2 + z^2 = R^2$. (In electromagnetism "flux" means the surface integral of the electric or magnetic field.)

8.204 Surface S consists of three surfaces: a disk of radius 1 on the xy-plane, a disk of radius 2 at $z = 3$, and a part of a cone that connects the two.



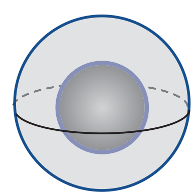
Vector $\vec{f}$ is $y^2\hat{i} + xe^z\hat{j} + \cos(xy)\hat{k}$. Find $\oiint_S \vec{f}\cdot d\vec{A}$.

8.205 Surface S consists of the cone and upper disk from Problem 8.204, but without the bottom disk.

(a) Vector $\vec{f}$ is $2x\hat{i} + 2y\hat{j} + 2z\hat{k}$. Evaluate $\iint_S\left(\vec{\nabla}\times\vec{f}\right)\cdot d\vec{A}$.

(b) Vector $\vec{g}$ is $\left(x^2 + e^y + \sqrt{z}\right)\hat{k}$. Evaluate $\iint_S\left(\vec{\nabla}\times\vec{g}\right)\cdot d\vec{A}$.

8.206 Region V is bounded between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 25$. The surface S (shown below) is the entire bounding surface of the region.



(a) If a vector field has positive flux through the inner sphere, which way would it have to point? Which way would it point to have positive flux through the outer sphere?

(b) Vector $\vec{f} = x^2\hat{j}$. Evaluate $\oiint_S \vec{f}\cdot d\vec{A}$.

(c) Vector $\vec{g} = (\ln r)\hat{r}$ in spherical coordinates. Evaluate $\iiint_V \left(\vec{\nabla} \cdot \vec{g}\right) dV$.

8.207 The surface S_1 is the cube with corners at $(0,0,0)$ and $(3,3,3)$, but it is missing the bottom (the side that would lie on the xy-plane). Your job is to add up the curl of $\vec{f} = e^z\hat{i} - 2xz\hat{k}$ along this most-of-a-cube.

(a) Describe and/or draw the curve C that bounds this surface.

(b) Describe and/or draw a simpler surface S_2 that is bounded by the same curve.

(c) Find $\iint_{S_1} \left(\vec{\nabla} \times \vec{f}\right) \cdot d\vec{A}$. You can do this any way you like, but it is easiest to use the results of your previous two answers!

In Problems 8.208–8.213 you'll be given a momentum density $\vec{V}$ for a fluid and something to calculate about it. Spend some time thinking about the easiest method for each one before jumping into calculations. Any letter other than the usual coordinate symbols is a constant.

8.208 $\vec{V} = ke^{(r-R)^2}\hat{r}$. Find the flow rate of the fluid out of a sphere of radius R centered on the origin.

8.209 $\vec{V} = (k/\rho)\hat{\rho}$. Find the flow rate of the fluid out of the portion of the sphere $x^2 + y^2 + z^2 \le R^2$ that lies in the first octant (meaning $x \ge 0$, $y \ge 0$, and $z \ge 0$).

8.210 $\vec{V} = \ln\left(x^2 - \sin x\right)\hat{i} + e^{-2\sqrt{y}}\hat{j} - 2y\hat{k}$. Find the circulation of the fluid around the ellipse $x^2 + 3y^2 = 1$, $z = 0$.

8.211 $\vec{V} = y\left(x^2 + y^2 - 1\right)\sin(x-3)\hat{i} - (x^2 + y^2 - 1)\hat{j} - 2\left(x^2 - y^2 + e^{xy}\right)\hat{k}$. Find the circulation of the fluid around a path that goes on the x-axis from the origin to $(1,0,0)$, from $(1,0,0)$ to $(0,1,0)$ along the curve $x^2 + y^2 = 1$ in the xy-plane, and finally back to the origin along the y-axis.

8.212 Find the line integral of $\left(x^2 + y^3\right)\hat{i} + 3xy^2\hat{j}$ from the origin to the point $x = -3\pi$, $y = 0$ along the path $\rho = \phi$ (where ρ and ϕ are the usual polar coordinates).

8.213 Find the flow rate of $\vec{V} = x\hat{i} + \left(\sin(\ln x) + y\right)\hat{j}$ through the cone $z^2 = 1 - x^2 - y^2$ in the region $z > 0$. *Hint*: You can start by finding the integral out of the surface made of the cone plus an added circle that closes off the bottom.

8.214 An important result in electrostatics is that inside a conductor, the electric field is everywhere zero. What does this say about the potential inside a conductor?

8.215 **The Hubble Flow** In 1929 Edwin Hubble discovered that the universe is expanding. From our point of view this means that all other galaxies are moving away from ours at a rate proportional to their distance from us: $\vec{v} = Hr\hat{r}$. We can to a reasonable approximation consider the density of galaxies to be the same value ρ throughout the observable universe, so the momentum density is $\vec{V} = \rho Hr\hat{r}$. (The variable ρ is commonly used to represent cylindrical coordinates, but it's also commonly used for density. Here we are using ρ for density, while r is the spherical coordinate.)

(a) Use the equation of continuity to derive an expression for the rate of change of the universe's density. Your answer should look like an ordinary differential equation for $\rho(t)$.

(b) If you assume the Hubble parameter H to be a constant, solve that differential equation to estimate how long ago the universe was twice its current density. Use the value $H = 2.4 \times 10^{-18}\ \text{s}^{-1}$.

(c) In reality H is not constant. Einstein's general theory of relativity predicts that for most of the history of the universe H has been approximately $2/(3t)$, where t is time since the big bang. Solve your differential equation for $\rho(t)$ again, using this function for H.

(d) Given that the current age of the universe is about 14 billion years, use your answer to Part (c) to give a more accurate answer for how long ago the universe was at twice its current density.