8.12 Additional Problems

In Problems 8.190–8.195 find all the field derivatives (gradient, divergence, curl, Laplacian) that can be calculated for the given fields, some of which are scalars and some of which are vectors.

8.190 \[ f = x^2 - y^2 \]
8.191 \[ \vec{f} = 3x^2i - 3xyj + (y - z)^2k \]
8.192 \[ f = (\rho/\zeta) \cos \phi \]
8.193 \[ \vec{f} = \rho \sin \phi \hat{\phi} + \rho \cos \phi \hat{\rho} + (\rho^2/\zeta) \hat{k} \]
8.194 \[ f = r^2 \sin \theta \cos \phi \]
8.195 \[ \vec{f} = r \sin \theta + r \cos \phi \hat{\phi} \]

In Problems 8.196–8.202 find a function \( F \) such that \( \nabla F = \vec{v} \) or prove that none exists. (This is one way of determining if \( \vec{v} \) is conservative. If it is, your \( F \) is \textit{negative} the potential function for \( \vec{v} \).)

8.196 \[ \vec{v} = 3i + 2k \]
8.197 \[ \vec{v} = xy i + xj \]
8.198 \[ \vec{v} = 3y i + 3yj + 3zk \]
8.199 \[ \vec{v} = e^z \left[ \sin(x + 2y) + \cos(x + 2y) \right] i + 2e^z \sin(x + 2y) j \]
8.200 \[ x \left( \sin(y)e^z i + \left( y^2 - x \cos(y)e^z \right) j \right) + (2z + x \sin(y)e^z) k \]
8.201 \[ \vec{v} = xe^{x+z}i + j + e^{x+z}k \]
8.202 \[ \vec{v} = (ye^{x+z} - 2(x + y + z)) i + (e^{x+z} - 2(x + y + z)) j + (e^{x+z} - (x + y + z)) k \]

Problems 8.203–8.207 ask questions that make no mention of the divergence theorem or Stokes’ theorem. But you can use these theorems to make many of these questions easier.

8.204 Surface \( S \) consists of three surfaces: a disk of radius 1 on the \( xy \)-plane, a disk of radius 2 at \( z = 3 \), and a part of a cone that connects the two.

\[ \vec{f} = y^2 i + xe^y j + \cos(xy)k \] Find \( \oint_S \vec{f} \cdot d\vec{A} \).

8.205 Surface \( S \) consists of the cone and upper disk from Problem 8.204, but without the bottom disk.

(a) Vector \( \vec{f} \) is \( 2x^2 i + 2yj + 2zk \). Evaluate \( \int_S (\vec{\nabla} \times \vec{f}) \cdot d\vec{A} \).

(b) Vector \( \vec{g} \) is \( \left( x^2 + e^y + \sqrt{z} \right) k \). Evaluate \( \int_S (\vec{\nabla} \times \vec{g}) \cdot d\vec{A} \).

8.206 Region \( V \) is bounded between the spheres \( x^2 + y^2 + z^2 = 9 \) and \( x^2 + y^2 + z^2 = 25 \). The surface \( S \) (shown below) is the entire bounding surface of the region.

(a) If a vector field has positive flux through the inner sphere, which way would it have to point? Which way would it point to have positive flux through the outer sphere?

(b) Vector \( \vec{f} = x^2 j \). Evaluate \( \oint_S \vec{f} \cdot d\vec{A} \).

Problems 8.203–8.207 ask questions that make no mention of the divergence theorem or Stokes’ theorem. But you can use these theorems to make many of these questions easier.
8.211 Find the line integral of \( \mathbf{V} = y \mathbf{i} + 2x \mathbf{j} \) from the origin to the point \( x = -3, y = 0 \) along the path \( \rho = \phi \) (where \( \rho \) and \( \phi \) are the usual polar coordinates).

8.212 Find the line integral of \( \mathbf{V} = x \mathbf{i} + 3y \mathbf{j} \) from the origin to the point \( x = 3, y = 0 \) along the path \( \rho = \phi \).

8.213 Find the flow rate of \( \mathbf{V} = x \mathbf{i} + (\sin(ln x) + y) \mathbf{j} \) through the cone \( z^2 = 1 - x^2 - y^2 \) in the region \( z > 0 \). \( \text{Hint:} \) You can start by finding the integral out of the surface made of the cone plus an added circle that closes off the bottom.

8.214 An important result in electrostatics is that inside a conductor, the electric field is everywhere zero. What does this say about the potential inside a conductor?

8.215 The Hubble Flow In 1929 Edwin Hubble discovered that the universe is expanding. From our point of view this means that all other galaxies are moving away from ours at a rate proportional to their distance from us: \( \mathbf{v} = H r \mathbf{r} \). We can to a reasonable approximation consider the density of galaxies to be the same value \( \rho \) throughout the observable universe, so the momentum density is \( \mathbf{V} = \rho H r \mathbf{r} \). (The variable \( \rho \) is commonly used to represent cylindrical coordinates, but it’s also commonly used for density. Here we are using \( \rho \) for density, while \( r \) is the spherical coordinate.)

(a) Use the equation of continuity to derive an expression for the rate of change of the universe’s density. Your answer should look like an ordinary differential equation for \( \rho(t) \).

(b) If you assume the Hubble parameter \( H \) to be a constant, solve that differential equation to estimate how long ago the universe was twice its current density. Use the value \( H = 2.4 \times 10^{-18} \text{ s}^{-1} \).

(c) In reality \( H \) is not constant. Einstein’s general theory of relativity predicts that for most of the history of the universe \( H \) has been approximately \( 2/(3 t) \), where \( t \) is time since the big bang. Solve your differential equation for \( \rho(t) \) again, using this function for \( H \).

(d) Given that the current age of the universe is about 14 billion years, use your answer to Part (c) to give a more accurate answer for how long ago the universe was at twice its current density.