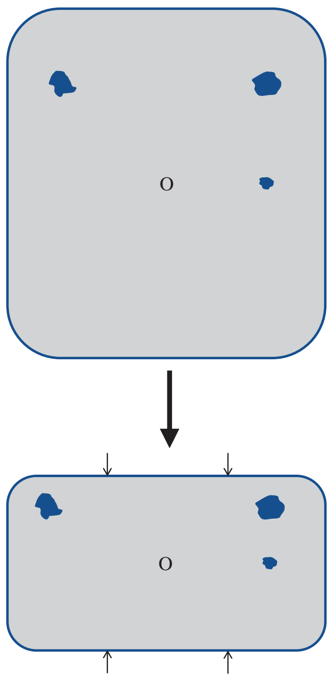
7.6 Additional Problems

7.137 The picture below shows a rubber sheet with three small stones on it. Their initial position vectors are $\hat{i}$, $\hat{i}+\hat{j}$, and $-\hat{i}+\hat{j}$, measured from the origin that's marked "O." The rubber sheet is compressed vertically until it is only half as tall as before, as shown below.

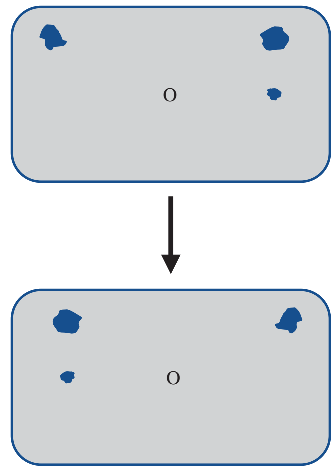


(a) What happens to the x-component of each vector? What happens to the y-component? First answer in words, then write equations that give the new x_1 and y_1 in terms of the old x_0 and y_0 for an arbitrary vector undergoing this transformation.

(b) What are the position vectors for the three stones after the transformation?

(c) Write a matrix that performs the transformation you described in Part (a). In other words, write a matrix that you can multiply by $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ to get $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Check your matrix by multiplying it by each of the initial rock positions to make sure it gives the final positions you predicted.

Next you look at a reflection of the rubber sheet in a mirror, so you see all the stones reversed left-to-right.



(d) What happens to the x-component of each vector? What happens to the y-component? First answer in words, then write equations that give the new x_2 and y_2 in terms of the old x_1 and y_1 for an arbitrary vector undergoing this transformation.

(e) What are the position vectors for the three stones after both transformations have been applied?

(f) Write a matrix that performs the transformation you described in Part (d). Check your matrix by multiplying it by each of the pre-reflection rock positions to make sure it gives the final positions you predicted.

Finally, consider the following transformation matrix.

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(g) Draw each of the following vectors, and then draw the vector you get after you multiply **M** by the given vector.

i. $\hat{i}$
ii. $\hat{j}$
iii. $-\hat{i}$
iv. $2\hat{i}+2\hat{j}$

(h) Looking at your drawings, what transformation does this matrix perform on vectors? You can't answer that it did such-and-such to this vector and so-and-so to that vector: you need one succinct statement of what effect it has on *all* vectors. (Equivalently you could say what happened to the rubber sheet, analogously to how we described the transformations above.) To test your answer, predict on visual terms where the vector $-\hat{j}$ will end up after this transformation and then multiply **M** by $-\hat{j}$ to check if you got it right.

(i) Multiply **M** by the previous positions of the three stones to find their final positions. Make sure they match what you would expect from your description of this matrix.

(j) Write a single matrix that takes the position vector of a stone, compresses it vertically by a factor of 2, reflects it horizontally, and then does whatever transformation **M** does. Check your matrix by multiplying it by each of the original rock positions to make sure it gives the final positions you calculated in Part (i).

(k) Write a matrix to return the stones from their final positions (after all three transformations) to their initial positions $\hat{i}$, $\hat{i}+\hat{j}$, and $-\hat{i}+\hat{j}$.

7.138 Matrix **A** rotates any point matrix 20° clockwise. Matrix **B** stretches any point matrix by a factor of 3 in the *x*-direction. *Nothing in this problem should require you to write or multiply any matrices.*

(a) What does matrix $\mathbf{A}^{-1}$ do to a matrix?

(b) What does matrix $\mathbf{B}^{-1}$ do to a matrix?

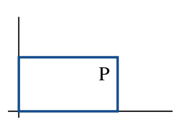
(c) What is the determinant $|\mathbf{A}|$?

(d) What is the determinant $|\mathbf{B}|$?

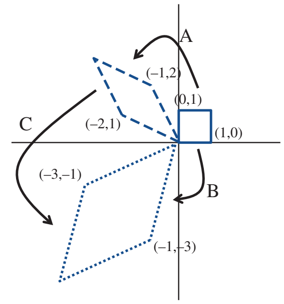
(e) What is the determinant $|\mathbf{B}^{-1}|$?

(f) What is the determinant $|\mathbf{AB}|$?

(g) Matrix **P** draws the rectangle shown below. Draw the shapes represented by point matrices **AP**, $\mathbf{A}^{-1}\mathbf{P}$, **ABP** and **BAP**.



7.139 In the figure below, matrix **A** transforms the solid square into the dashed diamond, and matrix **B** transforms the solid square into the dotted diamond.



(a) Matrix **A** turns the point $(0, 0)$ into $(0, 0)$ but that's no surprise—any matrix does that. **A** also turns the point $(1, 0)$ into $(-1, 2)$, and turns the point $(0, 1)$ into $(-2, 1)$. Finally it turns the point $(1, 1)$ into some point in the second quadrant. You can't quite see what that last point is—and that's OK. Using the points that you do know, find matrix **A**.

(b) Matrix **B** turns $(1, 0)$ into $(-3, -1)$, while $(0, 1)$ becomes $(-1, -3)$. Find matrix **B**.

(c) Matrix **C** transforms the dashed diamond into the dotted diamond. Write a symbolic matrix equation, using only the letters **A**, **B**, and **C** (no numbers!) that expresses the relationship between **C**, **B**, and **A**.

(d) Solve the equation symbolically to find matrix **C**.

(e) Calculate matrix **C**.

(f) The drawing shows two points for which we can clearly see what matrix **C** should do. Test your answer on those two points.

7.140 Write the matrices for rotating 2D shapes by an angle α and rotating them by an angle β. Multiply the two matrices and prove using trig identities that the resulting matrix rotates shapes by the angle $\alpha + \beta$.

7.141 Consider a computer animation package in which every object is represented by a $3 \times n$ point matrix: n points, each with x-, y-, and z-coordinates, that the computer will draw in order and connect with line segments. Stretches, reflections, and rotations, as well as composites of these operations, can be represented by matrix multiplications. But one of the simplest operations, a "translation"—retaining the shape and orientation of an object but changing its position in space—is harder in this representation.

(a) Object **A** comprises three points. (You can call the first point (x_1, y_1, z_1) and so on for the others.) Write a matrix operation that will add a constant X to its x-coordinates, Y to its y-coordinates, and Z to its z-coordinates. *Hint*: This operation will not be a matrix multiplication.

(b) Object **B** comprises four points. Write a matrix operation that will perform the same translation to this object.

You should have discovered that you needed a different matrix to perform this same operation on the two objects. This situation is not ideal for programming. Also, the fact that translation is not done by matrix multiplication makes it harder to form compound transformations made of translations and rotations. The solution is to represent every three-dimensional point with a four-dimensional vector, where the fourth coordinate is always a 1.

(c) Write the matrix that represents object **A** in this way. (It won't be $3 \times n$ anymore.)

(d) Multiply the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{pmatrix}$ by object **A**. What transformation does it perform?

(e) Does matrix **T** have the same effect on object **B**? How do you know?

7.142 If L is a line 70° counterclockwise from the x-axis then a reflection about L is accomplished by this matrix.

$$\mathbf{M} = \begin{pmatrix} \cos^2 70° - \sin^2 70° & 2\sin(70°)\cos(70°) \\ 2\sin(70°)\cos(70°) & \sin^2 70° - \cos^2 70° \end{pmatrix}$$

(a) Let $\vec{V} = \hat{i}$. Calculate $\mathbf{M}\vec{V}$. (Use decimals instead of carrying around all the sines and cosines.) Plot $\vec{V}$ and $\mathbf{M}\vec{V}$. If they don't appear reflected about L figure out what you did wrong.

(b) For a more rigorous check, you can verify that **M** has the right eigenvectors and eigenvalues. Without doing any calculations, in what directions should the eigenvectors of **M** point and what should their eigenvalues be?

(c) Find the eigenvectors in component form. This part has nothing to do with **M**. Just look at the line L and do the trig to find the vectors that point in the directions you identified. This time leave your answers in terms of trig functions, not as decimals.

(d) Verify that those vectors are eigenvectors of **M** with the eigenvalues you predicted.

7.143 Consider the set of matrices that stretch vectors by a factor λ_1 along the axis $y = x$ and by λ_2 along $y = -x$, for all real values of λ_1 and λ_2.

(a) Write such a matrix in terms of λ_1 and λ_2.

(b) Assuming addition and multiplication by a scalar are defined in the usual way for matrices, is this set of matrices a vector space?

(c) If you found that it is a vector space, what are its dimensions?

7.144 *[This problem depends on Problem 7.143.]* Assuming "addition" is defined by the usual rule of matrix multiplication and multiplication by a scalar is defined in the usual way for matrices, show that the set of matrices described in Problem 7.143 does *not* define a vector space. Given this addition rule, how would you define multiplication by a scalar so that this is a vector space? *Hint*: Look at what distributivity of scalar multiplication with respect to vector addition implies for the eigenvalues of $2\mathbf{a}$ with this new rule.

7.145 The space of all 2D rotation matrices $\mathcal{R}$ is a vector space, but not if you define matrix addition and multiplication by a scalar in the usual ways.

(a) Show that if you define addition and multiplication by a scalar in the way we usually do for matrices, $\mathcal{R}$ is not a vector space.

(b) Instead, you can define the sum of two rotation matrices as the rotation matrix for the sum of their two angles. (This is equivalent to multiplying the two matrices in the usual way.) You can similarly define multiplication by a scalar as multiplying the rotation angle by that scalar. With these definitions $\mathcal{R}$ is a vector space. What is its dimension?

7.146 The set of all rank 3 tensors is a vector space if addition and multiplication by a scalar are defined componentwise, as they are for matrices. Show that this vector space obeys distributivity of scalar multiplication with respect to vector addition.

7.147 **Inertia Tensor** In introductory physics you probably learned that a body has a "moment of inertia" I about any possible axis, which plays a role analogous to mass for linear motion. You may not have been taught that **I** is a tensor with nine components. (In this problem and the next we will use the

For Problem 7.149

	Cost per serving ($)	Calories	Sodium (mg)	Sugar (g)
Peanut butter	0.50	94	73	1.5
Salted Lentils	1	230	240	2
Spam	0.20	174	770	0
Soylent green	0.75	400	100	1

symbol **I** for the inertia tensor—no relation to the identity matrix.) For a continuous object of density ρ, $I_{ij} = \int_V \rho \left[\delta_{ij}\left(\sum_k x_k^2\right) - x_i x_j\right] dV$, where $\int_V dV$ means a triple integral over the volume, and x_1, x_2, and x_3 are the three Cartesian coordinates x, y, and z. The "Kronecker delta" δ_{ij} equals 1 if $i = j$ and 0 otherwise, so the sum over k only appears in the diagonal elements of **I**.

(a) Calculate the inertia tensor of a uniform cube of mass M with corners at the origin and the point (L, L, L).

(b) Angular momentum is given by the equation $\vec{L} = \mathbf{I}\vec{\omega}$. If the cube in Part (a) moves with angular velocity $\omega = a\hat{i} + b\hat{j}$ where a and b are constants, calculate its angular momentum.

7.148 *[This problem depends on Problem 7.147.]*

(a) The angular equivalent of Newton's second law is $\vec{\tau} = \mathbf{I}\vec{\alpha}$. Rewrite this equation to give $\vec{\alpha}$ as a function of $\vec{\tau}$ and then answer the equation: if a torque $\tau = \tau_0\hat{k}$ is applied to the cube in Problem 7.147, find the angular acceleration α of the cube.

(b) Find the inertia tensor of the same cube about a set of axes where the z-axis is unchanged and the x-axis goes diagonally through the bottom face of the cube. *Hint*: this can be done more easily with a rotation matrix than by integrating all over again.

(c) The "principal axes" of a body are the ones for which the inertia tensor is diagonal. Find the principal axes of this cube. (The origin remains unchanged at the corner of the cube. If the origin were in the center the principal axes could be guessed from symmetry.)

(d) If you apply a torque about the first of the principal axes you found, in what direction will the cube rotate?

7.149 **The Diet Problem** The company Sumptuous Land Of Plenty has just been awarded a contract to make stew for school lunches. Each serving must have between 600 and 700 calories, no more than 710 mg of sodium, and no more than 10 g of sugar.[11] The company has hired you to create the recipe that meets these requirements for the least possible cost, using the ingredients in the table above. Write, but do not solve, a simplex tableau to answer the question: how many servings of each ingredient should go in the recipe?

7.150 *[This problem depends on Problem 7.149.]* How many servings of each ingredient should go in the recipe?

7.151 **Exploration: The Schwarz Inequality.** The Schwarz inequality says that for any two vectors **a** and **b** in an inner product space:

$$|(\mathbf{a}, \mathbf{b})| \leq ||\mathbf{a}||\ ||\mathbf{b}|| \tag{7.6.1}$$

You'll prove this important inequality in this problem.

(a) First, let **a** and **b** be spatial vectors. Explain why the Schwarz inequality must hold in this case. What must be true of the two spatial vectors in order for the two sides of Equation 7.6.1 to be equal?

Now let **a** and **b** be generalized vectors. Define a new vector

$$\mathbf{c} = \mathbf{a} - \frac{(\mathbf{a}, \mathbf{b})}{||\mathbf{b}||^2}\mathbf{b}$$

(b) Show that **c** and **b** are orthogonal.

[11] The calorie and sodium requirements are from the "National School Lunch Program" guidelines for 6th–8th grade, and the nutrition information for peanut butter and spam comes from nutrition Web sites. Everything else in the problem we just made up.

(c) You can rewrite the definition of $\mathbf{c}$ as $\mathbf{a} = \mathbf{c} + \left[(\mathbf{a}, \mathbf{b})/||\mathbf{b}||^2\right] \mathbf{b}$. Use that equation for $\mathbf{a}$ to calculate $||\mathbf{a}||^2$. Use your result from Part (b) to simplify your answer as much as possible. *Hint*: Remember that $(\mathbf{a}, k\mathbf{b}) = k^*(\mathbf{a}, \mathbf{b})$.

(d) Rearrange your result to Part (c) to prove Schwarz's inequality.

(e) What must be true of $\mathbf{c}$ in order for Schwarz's inequality to be an equality? (Assume that $\mathbf{a}$ and $\mathbf{b}$ are both non-zero.)

(f) Show that Equation 7.6.1 is an equality only if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.