

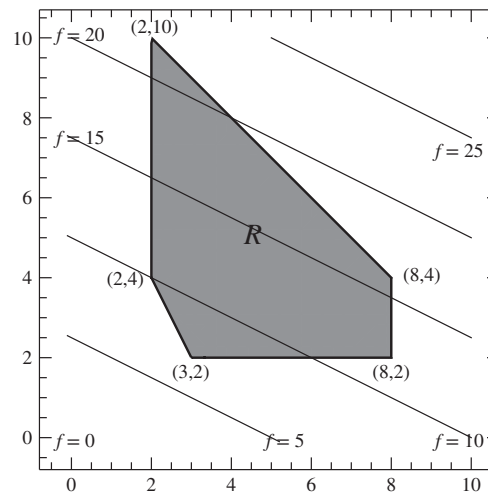
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## 7.5 Linear Programming and the Simplex Method

“Linear Programming” means optimizing a linear function subject to linear constraints. In principle this is about as easy as an optimization problem can get, but many problems involve so many variables that brute force methods are impractical. In this section we describe the “simplex algorithm,” a variant of row reduction that can handle problems with thousands of variables or more.

### 7.5.1 Discovery Exercise: Linear Programming and the Simplex Method

The picture shows a closed region  $R$  bounded by five lines. (The term “closed” indicates that the boundary lines are part of the region.) The picture also shows five contour lines of the function  $f(x, y) = x + 2y$ .



1. What point in region  $R$  has the largest possible value of  $f(x, y)$ ?
2. What point in region  $R$  has the smallest possible value of  $f(x, y)$ ?

*See Check Yourself #47 in Appendix L*

3. Redraw region  $R$  and then draw in four contour lines of the function  $g(x, y) = y - x$ .
4. What point in region  $R$  has the largest value of  $g$ ? The smallest value?

You should have found that all four extrema occurred at the vertices of region  $R$ . When we know in advance that this is going to be true, we can search for minima and maxima by looking only at the vertices.

5. Let  $h(x) = y + 2x$ . The minimum of  $h(x)$  within region  $R$  occurs at an infinite number of points. What points are those? Could we still find the maximum and minimum if we only looked at the vertices?
6. In many optimization problems the maximum or minimum is found inside the region, or along a border. Why did all the extrema in this exercise occur at the vertices? *Hint:* Your answer will involve a particular property of the functions  $f(x)$ ,  $g(x)$ , and  $h(x)$ , and also a property of the region  $R$ .



## 7.5.2 Explanation: Linear Programming and the Simplex Method

Chapter 4 presented two different methods of optimization, one based on the gradient and one on Lagrange multipliers. These techniques find critical points, either in the interior of the allowed region or on its boundaries. If you want to find a global maximum or minimum you evaluate the function at all of those critical points and at the endpoints of the boundaries.

In this section we will consider linear functions with linear constraints. There are never any critical points, so all you need to do is check the endpoints of the boundaries—the vertices. But when the problem involves hundreds or thousands of variables, an efficient algorithm for optimizing a function without checking every vertex becomes essential. Of course our sample problems will involve only a few variables. But if you imagine scheduling airplane flights or laying out a factory floor, you can see the importance of highly scalable algorithms.

### A Sample Problem: Two Machines

Your company has two machines. The cheap machine produces 2 items per hour and costs \$4 per hour to run. The fancy machine produces 4 items per hour and costs \$6 per hour to run. You can't run either machine for more than 40 hours per week and your total budget for running the machines is \$300. How many hours per week should you run each machine in order to maximize the number of items you produce?

As in any optimization problem you have an “objective function” that you want to maximize (or minimize) subject to constraints on the variables. In this case the objective function is the number of items you produce per week. The constraints include the three stated explicitly above, plus the implicit constraint that you can't run a machine for a negative amount of time.

$$\begin{aligned} \text{objective function: } f(F, C) &= 4F + 2C \\ \text{constraints: } F &\leq 40; \quad C \leq 40; \quad 6F + 4C \leq 300; \quad F, C \geq 0 \end{aligned} \quad (7.5.1)$$

What makes this a “linear programming” problem is that the objective function and the constraints are all based on linear functions. In this problem the constraints are all inequalities but they can also be equations as long as they are linear.

Very small problems can often be solved by inspection. In this example the fancy machine is more efficient so you'll want to run it for 40 hours and use the rest of your budget running the cheap machine. But things get complicated once you throw in maintenance costs, shipping times, different types of workers, raw materials needed, and so on. In this section we describe the “simplex algorithm,” the most commonly used method for solving linear programming problems.

### A Picture and Some Words

The independent variables in Equations 7.5.1 represent machine hours, but we can think of them as coordinates that define a two-dimensional space. The five constraints define a region in that space.

The Discovery Exercise (Section 7.5.1) was designed to convince you of the following fact.



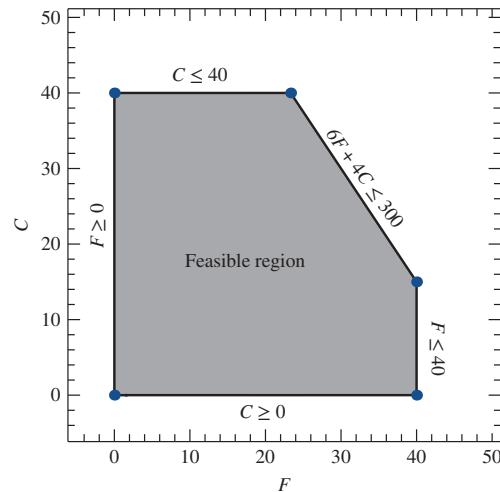
### Very Important Fact



If your objective is a linear function, and your constraints are linear functions that define a closed bounded region, then the minimum and maximum values of your objective function within the region occur at vertices of the boundary.



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**FIGURE 7.5** The “feasible region” for the machine problem. The blue dots at the vertices indicate “basic feasible solutions.”

A “feasible solution” of Equations 7.5.1 is any point in the shaded region of Figure 7.5; that is, any combination of the independent variables that satisfies all the constraints in the problem. A “basic feasible solution” is any vertex of the boundary of the region.

The important fact above suggests an optimization technique based on basic feasible solutions. Find all the vertices—that is, all the points of intersection of the constraints—and evaluate the objective function at each one. The highest value you find is the maximum value of the function within the feasible region.

That’s absolutely true, but it’s not practical. Consider the region bounded by the constraints  $x + 2y + 10z \leq 20$ ,  $8x + y + 4z \leq 12$ ,  $2x + 6y - z \leq 6$ ,  $3y + z \leq 3$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ . We’ll get you started by telling you that one of the vertices is the origin. Now you find the rest.

We trust you see the difficulty. With three variables each constraint is a plane, the feasible region is a polyhedron, and each vertex is the intersection of three constraints. With a hundred variables and a thousand constraints ... well, we need a systematic approach. Each step of the simplex method moves from one vertex to another vertex at which the objective function is higher. In cases where the origin is one of the vertices, you start there and step from vertex to vertex until you reach the maximum. In cases where the origin is not one of the vertices we add a “first phase” that finds a vertex to use as a starting point.

Before we present the method, we need to make a few more points about constraints.

- A constraint that is an *inequality* forms a boundary of the feasible region, and there is no limit to how many you can have. For instance, the feasible region for two variables with 17 inequality constraints would be a 17-sided polygon.
- A constraint that is an *equation* reduces the dimensionality of the problem by one. If you have three variables related by a linear equation, the feasible region lies on the plane defined by that equation. You could if you wished solve that equation to eliminate one variable and thus have a two-variable problem with only inequality constraints.
- If your objective function is parallel to a constraint then the maximum or minimum may occur along that constraint. (Imagine trying to minimize the function  $f(F, C) = C$  in Figure 7.5.) It still must be true that the extremum occurs at a vertex, but it can occur at more than one vertex and everywhere on the boundary connecting them.



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- If there is no combination of variables that satisfies the constraints the problem is “infeasible.” (Imagine trying to satisfy  $y < x + 3$  and  $y > 2x + 7$ , remembering that both  $x$  and  $y$  must be positive.) If there is a feasible region but the function has no maximum or minimum within it then the problem is “unbounded,” which can only occur if the feasible region is infinite. (See Problem 7.118.) All this looks simple in a 2D picture, but faced with a set of inequalities it may not be obvious if they bound a finite region, an infinite region, or no region at all. The simplex method, in addition to finding maxima and minima, helps us properly classify the region defined by the constraints.

### One Problem Written in Four Ways

The method we present here involves a gradual process of rewriting the initial problem. The objective function and the constraints will generally go through four distinct forms, two of which have names.

1. The original form in which they are presented (such as Equations 7.5.1 above).
2. The “normal form.”
3. The “restricted normal form.”
4. And finally, the form that we actually want.

We’re going to begin at the end, so you know what we’re looking for. The following is a maximization problem presented in Form 4. (This is a new problem; we’ll come back to our machinery example later.)

$$\begin{aligned}
 f(x_1, x_2, x_3, x_4) &= -2x_3 - 7x_4 + 20 \\
 x_1 + \quad + 7x_3 + 13x_4 &= 20 \\
 x_2 - 2x_3 + x_4 &= 8 \\
 x_1, x_2, x_3, x_4 &\geq 0
 \end{aligned}
 \tag{7.5.2}$$

Once your problem looks like Equations 7.5.2, you’re done. Here’s why.

The last constraint says none of the variables can be negative. Based on that you should be able to convince yourself that this particular objective function cannot possibly be bigger than 20. So if the objective function can reach 20 exactly—that is, if  $x_3 = x_4 = 0$  is compatible with the other constraints—then we have our maximum. Plug that in and the first two constraints immediately become  $x_1 = 20$  and  $x_2 = 8$ , both of which satisfy the last constraint, so we have a solution.

Our approach to any linear programming problem will be to make it look like Equations 7.5.2 and then read off the solution as we did above. So before we proceed, we urge you to consider the following question: what are the defining properties of Equations 7.5.2? That is, what characteristics of the objective function and the constraints enabled us to see the answer with no real calculations? Jot down your answers. A little later in this explanation we’ll give you our answers and you can see how yours compare.

You may also want to consider how these properties would change if we were minimizing instead of maximizing. From here on we will only consider maximization problems, but the same technique applies with minor modifications to minimization. See Problem 7.119.

### Normal Form and Restricted Normal Form

The first step in the simplex method is to write the problem in “normal form.”<sup>6</sup>

<sup>6</sup>Many sources define “normal form” without the constant  $c_{n+1}$  in the objective function. We include it because in the course of the simplex solution such a constant will get added to  $f$  even if it wasn’t there to start with.





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### Definition: Normal Form of a Linear Programming Problem

A linear programming problem written in “normal form” asks you to maximize an objective function of  $n$  variables,  $f = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n + c_{n+1}$ , subject to  $x_i \geq 0$  and the following  $m$  linear constraint equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

All the constants  $b_i$  must be non-negative.

Not every linear programming problem looks like the normal form described above, but every linear programming can be *put* into normal form with three tricks.

- Rewrite inequality constraints as equations by introducing extra variables. For example,  $6F + 4C \leq 300$  becomes  $6F + 4C + S = 300$ ,  $S \geq 0$ .  $S$  is called a “slack variable” because it takes up the slack between the left and right sides of the original inequality. If  $6F + 4C \geq 300$  write  $6F + 4C - S = 300$ ,  $S \geq 0$ ; this  $S$  is called a “surplus variable.”
- If any of the  $b_i$  are negative, multiply that constraint equation by  $-1$ .
- Finally, you might have a variable  $x$  that doesn’t have the constraint  $x \geq 0$ . If it has a different limit you can fix this easily by replacing it with a new variable. If  $x \geq -7$  let  $y = x + 7$  or if  $x \leq 0$  let  $y = -x$ . Rewrite your problem in terms of  $y$  instead of  $x$  and you have the constraint  $y \geq 0$ . If  $x$  isn’t constrained replace it with two variables:  $x = y - z$ . Now  $y, z \geq 0$  covers all possible values of  $x$ . See Problem 7.120.

Those rules leave an ambiguity if you have a constraint such as  $x \geq 2$ . Because  $x$  is already constrained to be positive you could write  $x - S = 2$ . Our recommendation, however, is to define a new variable  $y = x - 2$  which gives you only one variable ( $y$ ) and only one constraint ( $y \geq 0$ ) to work with.

For the machine example (Equations 7.5.1) we only need the first of these tricks to get to normal form.

$$\begin{aligned} \text{objective function: } f(F, C) &= 4F + 2C \\ \text{constraints: } F + S_1 &= 40; \quad C + S_2 = 40; \quad 6F + 4C + S_3 = 300; \quad F, C \geq 0 \end{aligned} \quad (7.5.3)$$

You probably won’t be shocked to hear that it is conventional to write these equations without the variables. The augmented matrix for a linear programming problem in normal form is called a “simplex tableau.” Notice that we pull the variables in the equation for  $f$  to the left side to make them line up with the other equations, so they change sign.

$$\begin{pmatrix} f & F & C & S_1 & S_2 & S_3 \\ 1 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 6 & 4 & 0 & 0 & 1 & 300 \end{pmatrix} \quad (7.5.4)$$

Switching from a list of equations to a matrix is not a change in form—it’s just a different way of writing the same form. Given our list of four forms above you might expect our next step to be converting the machine example from “normal form” to “restricted normal form.” In this case, however, we are already in restricted normal form. Let’s see what that means.




**Definition: Restricted Normal Form of a Linear Programming Problem**

A “basic variable” is one that has a positive coefficient in one of the constraint equations and zero coefficient in the objective function and all the other constraints. A linear programming problem is in “restricted normal form” if it is in normal form (as defined above) and every constraint has at least one basic variable in it.

Restricted normal form categorizes the variables into two types. In Equation 7.5.4 the basic variables are  $S_1$ ,  $S_2$ , and  $S_3$ . (You can easily identify a basic variable by looking for a column with one positive entry and the rest zero.) The other variables, which generally make up the objective function, are “non-basic.” In Equation 7.5.4 these are  $F$  and  $C$ .

When the tableau is in restricted normal form you can immediately read off one solution by setting all the non-basic variables to zero. This particular tableau suggests  $F = C = 0$ ,  $S_1 = S_2 = 40$ , and  $S_3 = 300$ . The corresponding value of the objective function comes from the first row, and in this case is  $f = 0$ .

It is a true but non-obvious result that the solution represented by a tableau in restricted normal form is a basic feasible solution, meaning it corresponds to a vertex of the feasible region.

But we do not yet have an optimal solution. The difference between Equation 7.5.4 (“restricted normal form”) and Equations 7.5.2 (“what we actually want”) is that the coefficients of the non-basic variables in the first row are negative. That means setting them to zero does not maximize the objective function. (Remember that we moved those coefficients to the left side of the equation. Equation 7.5.2 was optimal because the coefficients in the objective function were all negative, but that means they would appear as positive numbers in the first row of the simplex tableau.)

We can now fully answer our own question “What are the defining properties of Equations 7.5.2?” The answer is all the requirements for normal form, plus the added requirements for restricted normal form, plus one more: all the coefficients in the first row of the tableau must be zero or positive. When you have your equations in that form, you have your solution.

You may recall that as soon as we converted this particular problem into normal form it was in restricted normal form. In this Explanation we will only consider problems of that type. In Section 7.5.3 we will take up the question of how to convert normal form to restricted normal form in cases where you have to do so manually. But for now, we will continue with the machine problem.

### Finding the Optimal Solution

Equation 7.5.4 represents one vertex of the feasible region, but it is not the right vertex. The simplex method gives us a way of stepping from this vertex to a better one. Each such step brings us closer to the goal until we reach the maximum value of the objective function.

Here is another way of expressing the same goal. Equation 7.5.4 is in restricted normal form, but it has negative variables in the top row. The simplex method gives us a way of replacing one such variable with a zero, while a different variable in the top row goes from zero to a positive number. Each such step improves our tableau until it reaches the form that we want.

Either way you look at it, each step follows the rules of row reduction.

Look at the second column in Equation 7.5.4, the  $F$ . You see a  $-4$  at the top (which is why we want to change this), a 1, a 0, and a 6. We are going to turn  $F$  into a basic variable, meaning it will have a non-zero coefficient in only one constraint. That constraint (called the “pivot”) will be the second row. We want to make all other entries in the  $F$  column go to zero. So we add four times the second row to the first row, and we subtract six times the second row from the fourth row.



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$$\begin{array}{c} f \quad F \quad C \quad S_1 \quad S_2 \quad S_3 \\ \left( \begin{array}{cccccc|c} 1 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 6 & 4 & 0 & 0 & 1 & 300 \end{array} \right) \end{array} \rightarrow \begin{array}{c} f \quad F \quad C \quad S_1 \quad S_2 \quad S_3 \\ \left( \begin{array}{cccccc|c} 1 & 0 & -2 & 4 & 0 & 0 & 160 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{array} \right) \end{array} \quad (7.5.5)$$

Make sure you followed the two row operations we used! Then look at the results.  $F$  has become a basic variable, while  $S_1$  has become non-basic. We are still in restricted normal form so the entire tableau still represents a solution, which is now  $f = 160$ . That's an improvement over our previous solution, but the  $-2$  at the top of the  $C$  column means it's still not optimal.

But before we fix that last problem, let's look at two decisions we made in turning Equation 7.5.4 into Equation 7.5.5.

- *Why did we get rid of  $F$  first instead of  $C$ ?* It was mostly arbitrary. We could have chosen  $C$  instead, and we would have eventually reached the same solution we're going to reach this way. It is possible in principle, however, for the simplex method to enter an infinite loop, and you can avoid this with "Bland's rule": always choose the leftmost column with a negative coefficient in the first row.<sup>7</sup> The cycling problem almost never arises in practice and there are algorithms for choosing a column that will move you to the optimal solution as quickly as possible, but Bland's rule provides a good simple approach.
- *Why did we choose the second row as the pivot?* That is not arbitrary! We could have used the fourth row to eliminate  $F$  coefficients from all other rows, but we would have ended up with a negative number on the right side of the second row; we would no longer be in normal form. (Try it!) Avoid this problem with the "minimum ratio rule." Having chosen  $F$  as the pivot column, look at all the constraints that have positive entries in the  $F$  column (ignoring those with zero or negative entries). For each such constraint calculate the ratio of the rightmost coefficient (the constant) to the  $F$  coefficient. The pivot is the row with the smallest ratio. (If two rows are tied for smallest Bland's rule says to choose the topmost one.) In Equation 7.5.4 the second row had a ratio of  $40/1 = 40$  and the fourth row had a ratio of  $300/6 = 50$ , so we had to use the second.

To finish the problem, we pick up from Equation 7.5.5. Since  $C$  still has a negative coefficient on top we do a pivot somewhere on that column. The ratio for the third row is  $40/1 = 40$  and for the fourth row  $60/4 = 15$ , so we pivot about the fourth row.

$$\begin{array}{c} f \quad F \quad C \quad S_1 \quad S_2 \quad S_3 \\ \left( \begin{array}{cccccc|c} 1 & 0 & -2 & 4 & 0 & 0 & 160 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{array} \right) \end{array} \rightarrow \begin{array}{c} f \quad F \quad C \quad S_1 \quad S_2 \quad S_3 \\ \left( \begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1/2 & 190 \\ 0 & 1 & 0 & 1 & 0 & 0 & 40 \\ 0 & 0 & 0 & 3/2 & 1 & -1/4 & 25 \\ 0 & 0 & 4 & -6 & 0 & 1 & 60 \end{array} \right) \end{array} \quad (7.5.6)$$

The basic variables are now  $F$ ,  $C$ , and  $S_2$ . Since there are no negative numbers in the first row, we've found the optimal solution. Setting the non-basic variables to zero gives the solution  $F = 40$ ,  $C = 15$ , and  $f = 190$ . In other words, you should run the fancy machine for 40 hours and use the rest of your budget running the cheap machine. (We expected that, remember?)

Suppose there is a variable whose coefficient in the top row is negative and it doesn't have any positive coefficients in the constraints? You'll show in Problem 7.121 that this implies the problem is unbounded.

<sup>7</sup>Bland, Robert G. (May 1977). "New finite pivoting rules for the simplex method". *Mathematics of Operations Research* 2 (2): 103–107.


**EXAMPLE The Simplex Method**
**Problem:**

Maximize  $f = 2x_1 + 3x_2 - x_3$  subject to  $x_1 + 3x_2 - x_3 \leq 6$ ,  $2x_1 + x_2 + 2x_3 \leq 4$ ,  $x_1, x_2, x_3 \geq 0$ .

**Answer:**

First convert the inequalities to equations using slack variables:  $x_1 + 3x_2 - x_3 + S_1 = 6$ ,  $2x_1 + x_2 + 2x_3 + S_2 = 4$ . Next write the initial simplex tableau. Remember that the coefficients in  $f$  are being pulled to the left of the equation, so they all switch sign.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad x_3 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 & 6 \\ 0 & 2 & 1 & 2 & 0 & 1 & 4 \end{array} \right) \end{array}$$

This is in restricted normal form with  $S_1$  and  $S_2$  as basic variables. The  $x_3$  coefficient in the top row is already positive, so we need to do something about the  $x_1$  and  $x_2$  entries; following Bland's rule we choose  $x_1$ . (If you want the practice you could redo the problem starting with  $x_2$  and verify that you get to the same final answer.) The ratios of the constant to the  $x_1$  coefficients are  $6/1 = 6$  and  $4/2 = 2$ , so we use row 3 as the pivot. Add the third row to the first row and subtract  $1/2$  of the third row from the second row.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad x_3 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & 1 & 4 \\ 0 & 0 & 5/2 & -2 & 1 & -1/2 & 4 \\ 0 & 2 & 1 & 2 & 0 & 1 & 4 \end{array} \right) \end{array}$$

The next pivot column is  $x_2$ . The ratios are  $4/(5/2) = 8/5$  and  $4/1 = 4$ , so the second row has the smallest one this time. Add  $4/5$  of the second row to the first one and subtract  $2/5$  of the second row from the third one.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad x_3 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 0 & 7/5 & 4/5 & 3/5 & 36/5 \\ 0 & 0 & 5/2 & -2 & 1 & -1/2 & 4 \\ 0 & 2 & 0 & 14/5 & -2/5 & 6/5 & 12/5 \end{array} \right) \end{array}$$

Since all of the coefficients in the first row are positive we've reached the maximum value:  $f = 36/5$ . This value is obtained by setting all of the non-basic variables to zero, which turns the bottom two rows into  $(5/2)x_2 = 4$  and  $2x_1 = 12/5$ . So the maximum value occurs at  $x_1 = 6/5$ ,  $x_2 = 8/5$ ,  $x_3 = 0$ .

**Stepping Back**

The simplex method begins with a linear optimization problem and consists of the following steps to solve it.

1. Use the tricks described above to ensure that every independent variable has a constraint of the form  $x_i \geq 0$ .
2. Add slack and/or surplus variables to any inequalities to turn them into equations.







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3. If any constraint equations have negative constant terms, multiply them by  $-1$ . Your problem is now in normal form.
4. Move the variables in the objective function to the left side and write the objective function and all constraints in an augmented matrix, a.k.a. a “simplex tableau.”
5. *If your problem isn't in restricted normal form use the “two-phase” method described below to fix that. If the problem is infeasible then it will not be in restricted normal form and the two-phase method will fail in a way we describe below.*
6. Find the solution.
  - (a) If none of the variables have negative coefficients in the first row you've found the solution. Skip to Step 7. If there is a variable with a negative coefficient in the first row and no positive coefficients below it the problem is unbounded, and you should stop.
  - (b) Choose the leftmost negative coefficient in the first row to change into a basic variable.
  - (c) Consider all the positive coefficients in the column for that variable and use the minimum ratio rule to select which one to pivot about. In case of a tie use the topmost of the possible ones as your pivot.
  - (d) Use the techniques of row reduction to set all of the coefficients in your pivot column except the pivot itself to zero. This will make that variable basic, and it should make one of the other variables non-basic. (Occasionally this step will also make another variable basic as well. You can simply proceed as usual when this happens.)
  - (e) Return to Step 6a.
7. Once the problem is in restricted normal form with no negative coefficients in the first row, you're done. Set all the non-basic variables to zero and read off the values of the basic variables and the objective function.

There are a couple of mistakes to watch out for in this process.

- Remember that you have to bring all the non-constant terms in the objective function to the left side of the equation before you write the tableau, so their signs will flip.
- A tableau is not in restricted normal form just because each constraint has a variable that doesn't appear in any other constraint. You also have to get the coefficient of that variable in the objective function equal to zero.

To minimize a function with the simplex method you have two options. First, you can take the negative of your objective function and maximize that. Alternatively, you can use the simplex method exactly as described here except that you pivot around columns with *positive* entries in the first row, and you're done when all the non-basic variables have negative entries there. See Problem 7.119.

At this point you know how to do basic simplex problems. We encourage you to practice with a few problems, starting with Problem 7.108, and get comfortable with the process. Then come back and read Section 7.5.3 dealing with the “two-phase” simplex method.

### 7.5.3 Explanation: The Two-Phase Simplex Method

One thing you may not have noticed about our examples so far is that all the constraints (other than the ubiquitous “no negative variables”) were of the form  $a_1x_1 + a_2x_2 + \dots \leq b$  where  $b > 0$ . It's not hard to convince yourself in such a situation that the origin must be one of the vertices. The simplex method begins its vertex-hopping from there.

But replace one of those constraints with  $a_1x_1 + a_2x_2 + \dots \geq b$  or  $a_1x_1 + a_2x_2 + \dots = b$  (still with  $b > 0$ ) and the origin is no longer in the feasible region. That's a problem because the simplex method is designed to move from one vertex to another, so where does it start? Here





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we discuss a technique you can use to find one vertex—that is, one basic feasible solution. From there you can begin the simplex method.

Here is an alternative way of expressing the same idea. In our previous examples we put our problems into normal form and found that they were also in restricted normal form. Here we discuss the extra step that is required when that doesn't happen.

Remember that in restricted normal form every constraint contains at least one basic variable. That variable's coefficient is positive in that constraint and zero in all other equations. For instance, if your problem has four constraints then you need to rewrite it with at least four basic variables.

One tool for accomplishing that is the standard row reduction operations. Choose any four variables and attempt to make them basic by zeroing out all but one of their coefficients. If you end up with positive numbers in all the right places then your problem is in the right form. If not, choose four different variables and try again. Eventually you will find a feasible solution (or exhaust all possibilities and thus prove that none exists). But this needle-in-a-haystack approach is not practical with large numbers of variables.

We therefore introduce a new trick. This trick is designed to find one basic feasible solution: that is, to put a tableau into restricted normal form. That is the first phase of the “two-phase simplex method” and the second phase then proceeds as we described above.

Consider the following example.

$$\begin{aligned} \text{objective function: } f(x_1, x_2) &= 2x_1 - 3x_2 \\ \text{constraints: } x_1 + 2x_2 &\leq 2; \quad x_1 + x_2 \geq 1; \quad x_1, x_2 \geq 0 \end{aligned} \tag{7.5.7}$$

We can put this in normal form by introducing a slack variable in the first constraint and a surplus variable in the second one:  $x_1 + 2x_2 + S_1 = 2$ ,  $x_1 + x_2 - S_2 = 1$ . We can then write a simplex tableau.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & -2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right) \end{array}$$

This looks almost like restricted normal form, but it's not because the  $S_2$  entry is negative. This tableau therefore does not represent a feasible solution: if we set  $x_1 = x_2 = 0$  we get  $S_2 = -1$ , which is not allowed. We need to rewrite this tableau to have two non-negative basic variables.

We begin by defining an entirely new problem, complete with new variables and a new objective function.

1. Add one new variable to each constraint. This example has two constraints so we add  $z_1$  to the first and  $z_2$  to the second. Visually this means adding a new  $z_1$  column (with a 1 in the first constraint and 0 in all other rows), and a new  $z_2$  column (1 in the second constraint and 0 in all other rows).
2. Define a new objective function  $f_2 = -z_1 - z_2$ . Visually this means replacing the top row with a new row with 1s in the new columns.

So here is the new problem we will solve.

$$\begin{aligned} \text{objective function: } f_2 &= -z_1 - z_2 \\ \text{constraints: } x_1 + 2x_2 + S_1 + z_1 &= 2; \quad x_1 + x_2 - S_2 + z_2 = 1; \quad x_1, x_2, S_1, S_2, z_1, z_2 \geq 0 \end{aligned}$$

We are going to use the simplex method, just as we presented it before, to solve this new problem. But what will that buy us? Optimizing the  $f_2$  that we just made up doesn't optimize the  $f_1$  in the problem. But we'll see below that the solution to this new problem is a basic




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feasible solution to the original problem, which we can use as a starting point for solving that problem.

In this example our simplex tableau starts here.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \\ \left( \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{array} \right) \end{array}$$

We begin by making both  $z$  variables basic, so we subtract the second row from the first and then subtract the third row from the first.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \\ \left( \begin{array}{ccccccc} 1 & -2 & -3 & -1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{array} \right) \end{array}$$

We are now in restricted normal form (for this problem, not the original problem). To get rid of the  $-2x_1$  we pivot about the third row, because it has the lowest ratio.

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \\ \left( \begin{array}{ccccccc} 1 & 0 & -1 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \end{array} \right) \end{array}$$

Now get rid of the  $-x_2$ , pivoting about the second row. (Both ratios are the same so we chose the topmost of them.)

$$\begin{array}{ccccccc} f_2 & x_1 & x_2 & S_1 & S_2 & z_1 & z_2 \\ \left( \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -2 & -1 & 2 & 0 \end{array} \right) \end{array} \tag{7.5.8}$$

The first phase is now over. We find that  $f_2$  reaches a maximum value of zero when  $z_1 = z_2 = 0$ . (Given that we defined  $f_2 = -z_1 - z_2$  for non-negative  $z$ -values, this result was predictable from the outset.) But we're now going to return to the original problem, undoing the two steps that we used to go from that problem to this new one.

1. Eliminate both columns that represent the  $z$  variables.
2. Replace the top  $f_2$  row with the original  $f_1$  row from the problem.

$$\begin{array}{ccccccc} f & x_1 & x_2 & S_1 & S_2 \\ \left( \begin{array}{ccccccc} 1 & -2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 & 0 \end{array} \right) \end{array} \tag{7.5.9}$$

The point of the first phase was to rewrite the constraints (all rows but the top one). Equation 7.5.8 represented a valid solution to the  $f_2$  constraints with  $z_1 = z_2 = 0$ , so Equation 7.5.9 must represent a valid solution to the original constraints. And Equation 7.5.8 had two basic variables, so they should still be basic in our new  $f_1$  tableau... right?

Not quite.  $x_1$  and  $x_2$  appear in only one constraint each, but they are no longer basic variables because they also appear in the objective function. So you now have to take an extra step to make them zero in the top row. Both of your humble authors have vivid memories





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of forgetting this step when we first learned the simplex method, so we particularly caution you to look for it! We will add twice the third row to the first row, and subtract three times the second row from the first row, and then we will really be in restricted normal form.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 0 & -5 & -7 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 \end{array} \right) \end{array}$$

The tableau is now in restricted normal form and the standard simplex method can begin. Pivot about the second row in the  $S_1$  column, then pivot about the second row in the  $S_2$  column.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 5 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right), \quad \begin{array}{c} f \quad x_1 \quad x_2 \quad S_1 \quad S_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 7 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 2 \end{array} \right) \end{array} \end{array}$$

The optimal value of  $f$  is 4, and it occurs at  $x_1 = 2$ ,  $x_2 = 0$ .

**EXAMPLE** The Two-Phase Simplex Method

**Problem:**

Maximize  $f = 4x_1 - 2x_2$  subject to  $x_1 + x_2 \leq 5$ ,  $3x_1 + x_2 = 6$ ,  $x_1, x_2 \geq 0$ .

**Answer:**

Before you look at our solution, we encourage you to graph this problem on the  $x_1 x_2$ -plane and figure out the correct answer.

We add a slack variable to the first constraint to turn it into  $x_1 + x_2 + S_1 = 5$ , and we get the following simplex tableau.

$$\begin{array}{c} f \quad x_1 \quad x_2 \quad S_1 \\ \left( \begin{array}{cccc} 1 & -4 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 3 & 1 & 0 & 6 \end{array} \right) \end{array}$$

This is not in restricted normal form; it needs one more basic variable. You could try to get another one with row reduction. In this case if you tried with  $x_1$  it would work and if you tried with  $x_2$  it wouldn't. (Try!) With many more variables, however, it would be prohibitive to find a good set of basic variables by trial and error, so we'll use the two-phase simplex method.

*Phase 1:* Add a new variable to each constraint and maximize the function  $f_2 = -z_1 - z_2$ .

$$\begin{array}{c} f_2 \quad x_1 \quad x_2 \quad S_1 \quad z_1 \quad z_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 5 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{array} \right) \end{array}$$

The  $z_i$  are the basic variables, so to get this into reduced normal form we need to get their coefficients in the first row to zero. To do that we subtract the sum of the lower two rows from the first one.



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$$\begin{array}{cccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 \\ \left( \begin{array}{cccccc} 1 & -4 & -2 & -1 & 0 & 0 & -11 \\ 0 & 1 & 1 & 1 & 1 & 0 & 5 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{array} \right) \end{array}$$

After that we proceed with the usual simplex method. We pivot about the bottom row in the  $x_1$  column, and then we pivot about the middle row in the  $x_2$  column.

$$\begin{array}{cccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 \\ \left( \begin{array}{cccccc} 1 & 0 & -2/3 & -1 & 0 & 4/3 & -3 \\ 0 & 0 & 2/3 & 1 & 1 & -1/3 & 3 \\ 0 & 3 & 1 & 0 & 0 & 1 & 6 \end{array} \right), & \begin{array}{cccccc} f_2 & x_1 & x_2 & S_1 & z_1 & z_2 \\ \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2/3 & 1 & 1 & -1/3 & 3 \\ 0 & 3 & 0 & -3/2 & -3/2 & 3/2 & 3/2 \end{array} \right) \end{array} \end{array}$$

The first phase is over. The optimum solution  $f_2 = 0$  occurs when  $z_1 = z_2 = 0$ , as we hoped. We could also read off the values of the other variables, but we don't care since this isn't actually the problem we wanted to solve. What we care about is that we've turned two of the original variables into basic variables. So we continue...

*Phase 2: Toss out the  $z_i$  columns and put the original objective function back in.*

$$\begin{array}{cccc} f & x_1 & x_2 & S_1 \\ \left( \begin{array}{cccc} 1 & -4 & 2 & 0 & 0 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 0 & -3/2 & 3/2 \end{array} \right) \end{array}$$

Now comes the step we warned you about; we are not in restricted normal form! The variables  $x_1$  and  $x_2$  are no longer basic until we zero out their coefficients in the top row. That requires subtracting 3 times row 2 and adding  $4/3$  times row 3.

$$\begin{array}{cccc} f & x_1 & x_2 & S_1 \\ \left( \begin{array}{cccc} 1 & 0 & 0 & -5 & -7 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 0 & -3/2 & 3/2 \end{array} \right) \end{array}$$

The pivot column is the  $-5$  on top, and it only has one positive entry below it so we pivot about that 1.

$$\begin{array}{cccc} f & x_1 & x_2 & S_1 \\ \left( \begin{array}{cccc} 1 & 0 & 10/3 & 0 & 8 \\ 0 & 0 & 2/3 & 1 & 3 \\ 0 & 3 & 1 & 0 & 6 \end{array} \right) \end{array}$$

And at last we're done. The optimal solution is  $f = 8$  and it occurs at  $x_1 = 2$ ,  $x_2 = 0$ .

### Stepping Back

In the examples we solved in Section 7.5.2, all of the constraints were of the "less than or equal" form. Algebraically that meant that every constraint was written with a slack variable and the initial simplex tableau started in restricted normal form, with those slack variables as the initial set of basic variables. That initial basic feasible solution corresponded to setting



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all of the  $x_i$  equal to zero. In other words the initial vertex was the origin, and the simplex method proceeded from there.

In this section we looked at problems where the constraints included equations or “greater than” inequalities, which often means the origin is not in the feasible region. There’s no obvious starting vertex, which shows up algebraically in the fact that the initial simplex tableau isn’t in restricted normal form. The two-phase simplex method finds you a vertex to start from.

The outline of this process always looks the same.

1. Add one new variable  $z_i$  to each constraint. Define a new objective function  $f_2$  that is  $-1$  times the sum of all these new variables.
2. Create the simplex tableau for your new problem. Add the sum of all the constraint rows to the first row so that the new  $z$  variables are all basic.
3. Use the simplex method to maximize  $f_2$ .
4. Assuming all the  $z_i$  are non-basic, remove the  $z$  columns and put the original objective function back into the first row.
5. You should now have one variable in each constraint that doesn’t appear in any other constraint. Use row reduction techniques to set the first-row coefficients of those variables to zero. You are now in restricted normal form.
6. Use the simplex method to maximize  $f$ .

If the first phase doesn’t have an optimal solution where the  $z_i$  are all non-basic, that means the original problem was infeasible.

### 7.5.4 Problems: Linear Programming and the Simplex Method

**7.102** Equation 7.5.6 represents the final tableau for a linear optimization problem. Convert this matrix into an objective function and a set of constraints. Then explain using words and equations how you can find the optimal solution for the problem so expressed, and how you know your solution is optimal. *Hint:* we provide a similar explanation for the different problem represented by Equations 7.5.2.

For Problems 7.103–7.105 sketch the feasible region in the  $x_1x_2$ -plane. Add contour lines of the objective function to your sketch and use them to predict the vertex where  $f$  will be maximized. Then calculate  $f$  at each vertex and verify your answer.

**7.103**  $f = x_1 + x_2, 2x_1 + x_2 \leq 2$

**7.104**  $f = 2x_1 + 3x_2, x_1 + 2x_2 \leq 4, x_1 + x_2 \leq 3$

**7.105**  $f = x_1 - 4x_2, x_1 + 2x_2 \geq 4, x_1 + x_2 \leq 4$

**7.106** In the Explanation (Section 7.5.2) we claimed that a tableau in restricted normal form always corresponds to a vertex of the feasible region. To see why that’s true consider our machine example, with the constraints  $C \leq 40, F \leq 40,$

$4C + 6F \leq 300$ . The feasible region for this problem is shown in Figure 7.5. Each boundary of this region corresponds to one of the variables in Equations 7.5.3 being equal to zero. For example, the bottom edge has  $F = 0$  and the rightmost edge has  $S_1 = 0$ .

- (a) Copy this sketch.
- (b) For each vertex of the feasible region, identify which two variables equal zero.
- (c) Equation 7.5.4 represents a basic feasible solution to the machine problem. It tells us that if  $F = C = 0$ —the lower left-hand corner of Figure 7.5—then  $f = 0$ . Equation 7.5.5 steps from that tableau to a different tableau. Which vertex does that represent, and what solution? Equation 7.5.6 steps to a third tableau; which vertex does that represent, and what solution?

**7.107** Suppose a linear programming problem started with two variables  $x_1$  and  $x_2$  and 7 inequalities, not counting  $x_1, x_2 \geq 0$ . Including the slack and surplus variables, the problem in normal form would have 9 variables.



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(a) At each vertex, how many of those variables would equal zero? *Hint:* consider what a sketch of the feasible region would look like *before* introducing slack and surplus variables.

(b) If you had a simplex tableau in restricted normal form for this problem, how many basic (non-zero) variables would it have? How many non-basic variables would it have?

You should have concluded that a vertex of the original problem corresponds to the same condition as a tableau in restricted normal form. Now consider a problem that has three variables with  $x_1, x_2, x_3 \geq 0$ , plus an additional  $m$  constraints. Once again each boundary would correspond to one variable being zero, but now each boundary is a plane.

(c) How many variables equal zero at each vertex? *Hint:* you can answer this easily if you're good at picturing in 3D, but if you're not remember that each boundary is a linear equation in the original three variables. How many of those equations must be simultaneously satisfied to define a point?

(d) How many basic and non-basic variables would a tableau in restricted normal form for this problem have?

Once again you should have concluded that a tableau in restricted normal form corresponds to a set of variable values that occurs at a vertex of the original problem.

**7.108 Walk-Through: The Simplex Method.**

Maximize the function  $f = x_1 + 2x_2$  subject to the constraints  $x_1 - x_2 \leq 4$ ,  $x_1 + 3x_2 \leq 6$ , and  $x_1, x_2 \geq 0$ .

- (a) Write the constraints in normal form by adding a slack variable to the first two inequalities.
- (b) Write the initial simplex tableau. It will have one row for the objective function and two rows for the constraints. (The non-negativity conditions are assumed, and don't appear in the tableau.) Remember that the coefficients in  $f$  switch signs when you bring them to the left side of the equation.
- (c) You'll use the first negative entry in the first row as your pivot column. According

to the minimum ratio rule, which row in that column should you use for your pivot?

- (d) Pivot about that spot and write the resulting tableau.
- (e) What spot (column and row) should you use as your next pivot? How do you know?
- (f) Pivot about that spot and write the resulting tableau.
- (g) Your tableau should now indicate that the problem is solved. What feature(s) of the tableau let you know this? (If it doesn't indicate that the problem is solved you've made a mistake. Go back and find it.)
- (h) What is the maximum value of  $f$  and the values of  $x_1$  and  $x_2$  at which it occurs?

**7.109** [This problem depends on Problem 7.108.]

Sketch the feasible region defined by the constraints in Problem 7.108. Add contours of  $f$  to your plot, and use those contours to explain how you could know which vertex the maximum of  $f$  is on. Verify that your answer matches your answer to Problem 7.108.

For Problems 7.110–7.116 find the maximum value of the function  $f$  subject to the given constraints. Assume in each case that all of the independent variables are constrained to be non-negative. You may find it helpful to first work through Problem 7.108 as a model.

For the problems with two variables you can check your answers by drawing the feasible region and using contours of  $f$  to find the optimal vertex. We don't recommend that technique for three-variable problems unless you have extremely good 3D drawing skills, and even then we wouldn't recommend it for 4D problems.<sup>8</sup>

**7.110**  $f = 3x_1 - 2x_2, x_1 + x_2 \leq 4$

**7.111**  $f = -5x_1 + 2x_2, x_1 - x_2 \leq 4, -x_1 + 2x_2 \leq 2$

**7.112**  $f = 2x_1 + x_2, x_1 - 3x_2 \leq 4, -x_1 + 2x_2 \leq 2$

**7.113**  $f = -2x_1 + x_2, x_1 - 3x_2 \leq 4, -x_1 + 2x_2 \leq 2$

**7.114**  $f = x_1 + 3x_2, 2x_1 - x_2 \leq 3, -x_1 + 2x_2 \leq 1, x_1 + 2x_2 \leq 2$

**7.115**  $f = 2x_1 - x_2 + x_3, x_1 + x_2 + x_3 \leq 1, x_1 + 2x_2 + 3x_3 \leq 2$

**7.116**  $f = x_1 + 3x_4, x_1 + x_2 + x_3 + x_4 \leq 1, x_1 - 2x_2 - x_3 + 2x_4 \leq 1$

<sup>8</sup>But see <http://www.felderbooks.com/papers/4dplots.html> if you really want to.

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**7.117** Maximize  $f = x_1 - x_2$  subject to the constraints  $x_1 + 3x_2 \leq 10$ ,  $x_1 \geq 0$ ,  $x_2 \geq 2$ . (The last constraint makes this a bit different from the problems in the block above.)

**7.118** If all the variables in a linear programming problem are bounded above and below so the feasible region is finite, the problem is guaranteed to be bounded. The converse is not true; you can have a bounded problem even if some variables are unbounded. Consider  $x_1 - x_2 \leq 2$ ,  $-x_1 + x_2 \leq 1$ ,  $x_1, x_2 \geq 0$ .

- Sketch the feasible region. Explain how your sketch shows that  $x_1$  and  $x_2$  are not bounded from above.
- Add contour lines of  $f = x_1 - 2x_2$  to your sketch and use them to predict whether  $f$  has a maximum in the feasible region, and if so where.
- Use the simplex method to maximize  $f$  subject to these constraints and verify that the maximum occurs at the vertex you predicted.
- Copy your sketch of the feasible region and add contour lines of  $g = x_1 + 2x_2$ . Predict whether  $g$  has a maximum in the feasible region and if so where, and then use the simplex method to check your prediction.

**7.119** To minimize a function you can use the simplex method exactly as we've described it, except that you keep pivoting until there are no *positive* coefficients in the top row. Use this method to minimize the function  $f = x_1 - x_2$  subject to  $x_1 + 2x_2 \leq 5$ ,  $2x_1 - x_2 \leq 3$ , and of course  $x_1, x_2 \geq 0$ .

**7.120** Maximize the function  $f = x_1 - 2x_2$  subject to the constraints  $x_1 + x_2 \leq 2$ ,  $x_1 - x_2 \leq 1$ ,  $x_1 \geq 0$ . This looks a lot like the problems we've been solving but there's no explicit constraint on  $x_2$ . Because this is such a small problem you could draw the feasible region and figure out the limits on  $x_2$ , but for large problems it's not always easy to figure out if a variable is bounded, never mind what its limits are. The systematic way to do this is to define  $x_2 = y - z$ , subject to  $y, z \geq 0$ . Rewrite the objective function and constraints in terms of  $y$  and  $z$  and then use the simplex method to maximize  $f$ . *Hint:* At the end you may have to make some arbitrary-looking decisions about  $y$  and  $z$ . Remember that what really matters is what they imply about  $x_2$ .

**7.121** Consider the following simplex tableau in restricted normal form.

$$\begin{array}{cccccc} f & x_1 & x_2 & S_1 & S_2 & S_3 \\ \hline 1 & 2 & -1 & 0 & 0 & 0 & 2 \\ 0 & 3 & -2 & 1 & 0 & 0 & 3 \\ 0 & -2 & -3 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}$$

This does not represent an optimal solution because the coefficient of  $x_2$  in  $f$  is negative. But there is no row in that column you can use as a pivot because none of the coefficients are positive. We claimed in the Explanation (Section 7.5.2) that this means the problem is unbounded, but we didn't explain why.

- Instead of setting  $x_1 = x_2 = 0$  set  $x_1 = 0$ ,  $x_2 = 1$ . Find the values of  $S_1$ ,  $S_2$ , and  $S_3$  that satisfy all of the constraints, and find the corresponding value of  $f$ .
- Repeat Part (a) but set  $x_2 = 100$ .
- What happens to  $f$  in the limit where you increase  $x_2$  without bound, adjusting the  $S$  variables to keep the constraints satisfied?
- Change the last entry in the  $x_2$  column from 0 to 1. Now try setting  $x_2 = 100$  as you did before and explain why it doesn't work.

**7.122 Walk-Through: The Two-Phase Simplex Method.** In this problem you will maximize the function  $f = 2x_1 - 3x_2$  subject to the constraints  $x_1 + x_2 \leq 4$ ,  $-x_1 + x_2 \leq 1$ ,  $x_1 + 3x_2 = 6$ ,  $x_1 \geq 0$ , and  $x_2 \geq 0$ .

- Begin by rewriting the inequality constraints as equations. Because both of them are "less-than" inequalities, both will introduce slack variables.
- Put the problem into a simplex tableau. As always this requires pulling the objective function variables to the left side (with the  $f$ ), thus changing all the signs.
- How can you tell that your tableau is not in restricted normal form?
- It is therefore time for the first phase. Your tableau represents three constraints so add three new variables:  $z_1$  to the first constraint,  $z_2$  to the second and  $z_3$  to the third. (Visually that adds three columns to the tableau, each with one 1 and two 0s.) You will also temporarily abandon your original objective function and set out to maximize a new function,  $f_2 = -z_1 - z_2 - z_3$ . (Visually this replaces



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the first row with a different one that represents  $f_2 + z_1 + z_2 + z_3 = 0$ .) Write the simplex tableau for starting the first phase.

- (e) Make the  $z$  variables basic by subtracting every constraint—the second, third, and fourth rows—from the first row.
- (f) Use the simplex method to optimize your  $f_2$  function.
- (g) Starting from the final tableau of the first phase, toss out the  $z$  columns and put back in the original objective function: that is, replace this top row with the original top row.
- (h) The variables that were basic at the end of the first phase may not be basic any more, because they appear in the top row. Make these variables truly basic by using the constraint rows to eliminate them from the objective row.
- (i) Use the simplex algorithm in the usual way to finish this problem. (This is the “second phase.”)

- 7.123** [This problem depends on Problem 7.122.] Sketch the feasible region defined by the constraints in Problem 7.122. Add contours of  $f$  to your plot, and use those contours to explain how you could know which vertex the maximum of  $f$  is on. Verify that your answer matches your answer to Problem 7.122.

For Problems 7.124–7.130 use the two-phase simplex method to find the maximum value of the function  $f$  subject to the given constraints. Assume in each case that all of the independent variables are constrained to be non-negative. You may find it helpful to first work through Problem 7.122 as a model. For the problems with two variables you can check your answers by drawing the feasible region and using contours of  $f$  to find the optimal vertex.

**7.124**  $f = 3x_1 - x_2$ ,  $x_1 + 2x_2 \geq 1$ ,  $3x_1 + 2x_2 \leq 5$

**7.125**  $f = -x_1 + 2x_2$ ,  $2x_1 + 3x_2 \leq 8$ ,  
 $x_1 - x_2 \leq 1$ ,  $-x_1 + 4x_2 = 3$

**7.126**  $f = 3x_1 - x_2$ ,  $x_1 - x_2 \geq 2$ ,  $x_1 + 2x_2 \leq 5$

**7.127**  $f = 2x_1 + x_2$ ,  $x_1 + x_2 \geq 2$ ,  $3x_1 + 2x_2 \leq 7$ ,  $x_1 \leq 1$

**7.128**  $f = 2x_1 - x_2 + x_3$ ,  $1 \leq x_1 + x_2 + 2x_3 \leq 2$

**7.129**  $f = 2x_1 - x_2 + 4x_3$ ,  $x_1 - x_2 + 2x_3 \geq 2$ ,  
 $x_1 + 3x_2 \leq 4$ ,  $x_3 \leq 1$

**7.130**  $f = x_1 - 2x_2 + x_4$ ,  $x_1 + x_2 + 2x_3 + 3x_4 \geq 1$ ,  
 $x_1 + x_4 \leq 10$ ,  $x_1 + 2x_2 - x_3 - x_4 = 6$

- 7.131** Maximize  $f = x_1 + 2x_2$  subject to the constraints  $2x_1 + 3x_2 \leq 28$ ,  $2x_1 + x_2 \geq 8$ ,  $2 \leq x_1 \leq 8$ ,  $x_2 \geq -2$ . (Note that—unlike in the problems above—we cannot assume here that  $x_2$  is a non-negative number.)

- 7.132 The Transportation Problem** Your warehouse in Atlanta has 300,000 nails, the one in New York has 200,000 and the one in Boston has 500,000. The stores in Chicago, St. Louis, and Louisville need 400,000, 300,000, and 300,000 nails respectively. Write, but do not solve, a simplex tableau to answer the question: how many nails should each warehouse send to each store? Assume the transportation cost is proportional to the distance, which is given below.

	Atlanta	New York	Boston
Chicago	700	800	1000
St. Louis	500	1000	1200
Louisville	400	800	1000

- 7.133** [This problem depends on Problem 7.132.]

How many nails should each warehouse send to each store?

- 7.134 The Assignment Problem**<sup>9</sup> Suppose you have three workers that each need to be assigned to a job. The workers have different levels of skill and experience; the cost of employing each one to do each job is given below. Define a set of variables  $x_{ij}$ , equal to one if worker  $i$  has job  $j$ , and zero otherwise. Write, but do not solve, a simplex tableau to minimize the total cost subject to the constraints that each worker has exactly one job and each job has exactly one worker.

	Job 1	Job 2	Job 3
Worker 1	1	3	2
Worker 2	2	3	5
Worker 3	2	2	4

- 7.135** [This problem depends on Problem 7.134.] Which worker should have each job?

- 7.136 Exploration: Moving Sand** In 1781 Gaspard Monge published his work on the “soil transport” problem: how to most efficiently

<sup>9</sup>The simplex method is not the most efficient way to solve the assignment problem for large numbers of variables. See, e.g., H.W. Kuhn, The Hungarian Method for the Assignment Problem, Naval Research Logistics Quarterly 2 (1955) 83–97.

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move a pile of dirt from a given starting shape to a given final shape. This problem has applications in many areas, including calculating distance between states of a quantum mechanical system.<sup>10</sup> As an example, suppose you have a square sandbox that goes from the origin to the point  $(5, 5)$ . Initially the height of the sand is given by  $S_0 = e^{-x^2-y^2}$ . You want to move it to a different corner:  $S_f = e^{-(5-x)^2-y^2}$ . The cost of moving a unit volume of sand is equal to the distance you move it. In this form the problem involves integrals over the initial and final distributions, but you can turn it into a linear programming problem by breaking the grid into discrete boxes. If each box is  $1 \times 1$  then you'll have a total of 25 boxes  $B_{ij}$ , where  $i$  and  $j$  each go from 0 to 4.

- (a) Make tables of the average values of  $S_0$  and  $S_f$  in each of the 25 boxes. *Hint:* once you calculate this for  $S_0$  you can get it for  $S_f$  from symmetry.
- (b) Your independent variables are  $x_{ijkl}$ , representing the amount of sand moved from  $B_{ij}$  to  $B_{kl}$ . Define an objective function representing the total cost of movement as a function of these variables.

Recall that cost is equal to distance times amount of sand. Naively you have  $5^4$  independent variables, but you can significantly reduce that number by tossing out any term that involves moving sand from a location where  $S_f \geq S_0$  or moving sand to a location where  $S_0 \geq S_f$ .

- (c) Define a set of constraints that reflects the fact that the amount of sand taken from each box equals  $S_0 - S_f$  (provided this is positive).
- (d) Define a set of constraints that reflects the fact that the amount of sand moved to each box equals  $S_f - S_0$  (provided this is positive).
- (e) Either write a simplex program or use an existing one to find the minimum cost to move the sand.
- (f) Look at the values of the  $x_{ijkl}$  in your final solution and describe in words how the sand was moved. You should find that the answer was predictable.
- (g) Find the minimal sand-moving cost for the same initial sand distribution, but with  $S_f = (1/2)e^{-(x^2+y^2)/4}$ .

<sup>10</sup>Karol Zyczkowski and Wojciech Słomczyński, "The Monge Distance Between Quantal States," J. Phys. A: Math. Gen. 31 (1998) 9095–9104.