CHAPTER 7

# Linear Algebra II (Online)

## 7.4 Row Reduction

Given a system of $n$ linear equations with $n$ unknowns, a determinant of zero tells you "this system has either no solutions or infinitely many solutions," but it doesn't tell you which situation you're in. A non-zero determinant means "this system has one unique solution," but it doesn't tell you what that solution is. Moreover, calculating determinants for very large matrices is computationally expensive. In this section we introduce a technique that fills in those holes: it tells you the solution, or it tells you that the equations are inconsistent, or it tells you that the equations are linearly dependent, and it's more efficient than finding a determinant for large systems of equations. In that sense, this section finishes the job that Section 6.7 started.

But in another sense, this section is independent of the rest of what we've been doing with linear algebra. In this section you will not multiply matrices. You will not find their inverses, determinants, or eigenvalues. The matrix manipulation rules in this section are unrelated to the rules we presented throughout Chapter 6. This is simply another way you can use grids of numbers to help you solve linear equations.

### 7.4.1 Explanation: Row Reduction

#### Solving Linear Equations by Elimination

You may have learned at some point to solve simultaneous linear equations using a technique called "elimination" or "addition and subtraction." You are always allowed to add or subtract two equations; in this technique you do so to isolate variables. We will demonstrate this process on the following three equations. The variables are $x$, $y$, and $z$; we use the letters $A$, $B$, and $C$ to refer to *equations*.

$$\begin{aligned} A_0 &: \; x - y + 2z = 10 \\ B_0 &: \; 3x - 2z = 11 \\ C_0 &: \; 2x + 4y - 6z = -6 \end{aligned} \tag{7.4.1}$$

First we eliminate $x$ from the bottom two equations by subtracting $3A_0$ from $B_0$ and $2A_0$ from $C_0$.

$$\begin{aligned} A_1 &: \; x - y + 2z = 10 \\ B_1 &: \; 3y - 8z = -19 \\ C_1 &: \; 6y - 10z = -26 \end{aligned}$$

Next we eliminate $y$ from the third equation by subtracting $2B_1$ from it.

$$\begin{aligned} A_2 &: \; x - y + 2z = 10 \\ B_2 &: \; 3y - 8z = -19 \\ C_2 &: \; 6z = 12 \end{aligned}$$

Dividing $C_2$ by 6 gives $z = 2$. You can plug this into $B_2$ to get $y = -1$ and plug both of these into $A_2$ to get $x = 5$.

### Row Reduction

In Chapter 6 we wrote the "matrix of coefficients" for $n$ equations with $n$ unknowns. That was a square matrix because it contained the coefficients of all the variables but ignored the constant terms. When we give the constants their own column, the resulting "augmented matrix" represents the entire system of equations.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 3 & 0 & -2 & 11 \\ 2 & 4 & -6 & -6 \end{pmatrix} \tag{7.4.2}$$

Equation 7.4.2 is perfectly equivalent to Equations 7.4.1; we just don't bother mentioning the variables because we know where they go. For instance, the row $\begin{pmatrix} 1 & -1 & 2 & 10 \end{pmatrix}$ represents the equation $x - y + 2z = 10$. Below we demonstrate "row reduction" on Equation 7.4.2; your job is to follow how this technique is line-by-line identical to the "elimination" example above. For instance, where we previously subtracted three times the first equation from the second equation, we now subtract three times the first *row* from the second row.

**EXAMPLE** **Row Reduction**

**Question:** Solve the three equations represented by Equation 7.4.2.

**Solution:**
Subtract three times the first row from the second row, and also subtract two times the first row from the third. This turns the first number in the last two rows into zero; that is, it eliminates $x$ from the second and third equations.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 0 & 3 & -8 & -19 \\ 0 & 6 & -10 & -26 \end{pmatrix}$$

Subtract two times the second row from the third one to get 0 in the first *two* spots of the third row.

$$\begin{pmatrix} 1 & -1 & 2 & 10 \\ 0 & 3 & -8 & -19 \\ 0 & 0 & 6 & 12 \end{pmatrix}$$

Divide the third row by 6 to get (0 0 1 2), which stands for $z = 2$. Add eight times that to the second row to get $\begin{pmatrix} 0 & 3 & 0 & -3 \end{pmatrix}$ and then divide that by 3 to get $y = -1$. Finally add the second row minus two times the third row to the first one, and our augmented matrix now looks like this.

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \tag{7.4.3}$$

We interpret this matrix as $x = 5$, $y = -1$, and $z = 2$: the same result we found above.

Now that we've done it, what did we do? Every step in row reduction consists of one of the three operations described below.

**Allowed Operations in Row Reduction**

1. Add a multiple of one row to another row.
2. Multiply a row by a non-zero constant.
3. Switch two rows.

These are called the "elementary operations" on a matrix.

(Switching rows is never necessary, but in some cases it makes the calculations easier.)

The goal is to keep performing elementary operations until the matrix looks something like Equation 7.4.3: a square identity matrix with an additional column on the right. Such a matrix gives all the answers directly, such as $x = 5$ and so on.

### Row Reduction with Dependent or Inconsistent Equations

In Chapter 6 we said that if the determinant of the matrix of coefficients is non-zero, the equations have a unique solution. If the determinant is zero, the equations are either linearly dependent (infinitely many solutions) or inconsistent (no solutions). In either case row reduction provides more information.

As an example, consider the following equations:

$$\begin{aligned} x - 2y + 5z &= 2 \\ 3x + y &= 9 \\ x - 9y + 20z &= -1 \end{aligned}$$

We begin merrily row reducing.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 3 & 1 & 0 & 9 \\ 1 & -9 & 20 & -1 \end{pmatrix}$$ Write an augmented matrix to represent the equations.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 7 & -15 & 3 \\ 0 & -7 & 15 & -3 \end{pmatrix}$$ Subtract three times the first row from the second, and subtract the first row from the third.

$$\begin{pmatrix} 1 & -2 & 5 & 2 \\ 0 & 7 & -15 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Add the second row to the third, and… uh-oh.

We've just found that our original equations are equivalent to the three equations $x - 2y + 5z = 2$, $7y - 15z = 3$, and $0 = 0$. That last row tells us that our original equations were linearly dependent; we thought we had three constraints, but we really only had two. If we want to find the values of $x$, $y$, and $z$, we're going to need more information.

If we had ended up with $0 = 7$ instead of $0 = 0$, that would have told us something quite different. Zero never equals seven, even on a very bad day. Such a result would indicate that our original equations were inconsistent: they had no solution.

## Rank of a Matrix

We have now seen enough about row reduction to classify a set of equations into one of three categories: "linearly dependent" (an infinite number of solutions), "inconsistent" (no solution), or "has a unique solution" (which row reduction then gives us). But there is more to know, and sometimes it's important to know it.

Suppose you start with twenty equations with twenty unknowns and discover they are linearly dependent. That means you cannot solve for all twenty variables because there is no unique solution. What you can do is specify some of the variables—either arbitrarily, or based on conditions outside your twenty equations—and then solve for the rest. But how many? Perhaps you can specify one variable and then solve for the other nineteen. On the other hand, perhaps you need to specify thirteen of the variables before you can find the remaining seven. This important distinction is captured in the "rank" of the augmented matrix.

**Definition and Use: Rank of a Matrix**

The "rank" of a matrix is its number of linearly independent rows.

If an augmented matrix of rank $R$ represents a set of equations, you can solve those equations to find $R$ of the variables in terms of the remaining variables.

In our above example, if your augmented matrix turned out to be rank twelve, you could solve for twelve of the variables in terms of the remaining eight.

Now you know what to do with the rank of a matrix, but how do you find it? The answer is, once again, row reduction. You keep manipulating until the matrix is in "row echelon form."

**Definition and Use: Row Echelon Form**

The "leading zeroes" in a matrix row are the zeroes on the left before any non-zero term. For instance the row (0 0 5 0) has two leading zeroes.

To see if a matrix is in "row echelon form" start at the top and look at each row in succession.

1. Each row must have more leading zeroes than the previous row until you reach a row that is all zeroes.
2. If you do reach a row that is all zeroes, all the rows below it must also have all zeroes. (In other words the all-zero rows are clustered at the bottom.)

When a matrix is in row echelon form, the number of rows that are *not* all zeroes is the rank of the matrix.

In the particular case of $n$ equations with $n$ unknowns, the augmented matrix has dimensions $n \times (n+1)$. In row echelon form each row must have more leading zeroes than the previous row, so the bottom row must have at least $n-1$ leading zeroes. If that bottom row does not consist of all zeroes then your matrix is rank $n$ and your equations are not linearly dependent. (They may or may not be inconsistent.)

EXAMPLE **Rank of a Matrix**

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & -3 & 2 & 0 & 7 \\ 0 & 2 & \pi & 4 & 9 & 0 \\ 0 & -6 & -3\pi & -12 & -27 & 0 \\ 0 & 0 & 0 & 4 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 5 & -3 & 2 & 0 & 7 \\ 0 & 2 & \pi & 4 & 9 & 0 \\ 0 & 0 & 0 & -12 & -27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix **A** is not in row echelon form, because the third row does not have more leading zeroes than the second. More row reduction is required before you can determine the rank of this matrix.

Matrix **B** is in row echelon form. It has three non-zero rows, indicating a rank three matrix. If this matrix represents a set of five equations, you could solve for three of the variables in terms of the other two.

### Summary: Rank of a Matrix

What does the rank of a matrix tell you about a system of equations? Before we answer that, remember that there are two ways to represent a system of equations in a matrix. The "matrix of coefficients" includes all the terms that involve the variables. The "augmented matrix" adds an additional column for the constant terms in the equations. We will consider both matrices in the following discussion—don't get them confused!

For $n$ equations with $n$ unknowns here are the possibilities.

**I.** *Row reduction produces a unique solution.*
- The equations are linearly independent and consistent.
- The matrix of coefficients has a non-zero determinant.
- The matrix of coefficients and the augmented matrix are both of rank $n$.

**II.** *Row reduction produces a row with all zeroes except the last column.*
- The equations are inconsistent (no solution).
- The matrix of coefficients has a zero determinant.
- The rank of the matrix of coefficients is less than $n$.
- The rank of the augmented matrix is larger than the rank of the matrix of coefficients.

**III.** *Row reduction produces $m$ rows with all zeroes (but none with all zeroes except the last column).*
- The equations are linearly dependent (infinitely many solutions). Specifically, only $n - m$ of the equations are linearly independent.
- The matrix of coefficients has a zero determinant.
- The matrix of coefficients and the augmented matrix both have rank $n - m$.
- You need to specify $m$ of the variables before you can solve for the remaining $n - m$ of them.

What if the augmented matrix has a lower rank than the matrix of coefficients? Take a moment to convince yourself that it's impossible; if this did happen it would mean that you had a matrix with all linearly independent rows, but adding a column to it made those rows linearly dependent.

Finally, these results can be generalized to $n$ linear equations with $u$ unknowns. Let $M$ be the matrix of coefficients and $A$ the augmented matrix, and let $R_M$ and $R_A$ be their ranks. The only possibilities are:

- If $R_A > R_M$ the equations are inconsistent.
- If $R_A = R_M = u$ the equations have one unique solution.

- If $R_A = R_M < u$ the equations have infinitely many solutions and can only be solved for $R_A$ of the unknowns in terms of the remaining ones.

In the last case listed above the $R_A$ unknowns that you solve for are called the "dependent variables" and the remaining ones that you have to specify are called the "free variables." You're guaranteed in this case that there is at least one set of $R_A$ variables that you can designate as dependent, but you cannot always do that for any set of $R_A$ variables that you choose.

## 7.4.2 Problems: Row Reduction

**7.85 Walk-Through: Row Reduction.** Consider the following equations:

$$
\begin{aligned}
2x + 3y + z + t &= 2 \\
-2x - y + 2z + 4t &= -21 \\
8x + 20y + 7z + 4t &= 10 \\
2x + 15y + 10z + 11t &= -34
\end{aligned}
$$

**(a)** Write the augmented matrix for this set of equations. (This will be a 4×5 matrix.)

**(b)** Replace the second row with the sum of the first and second rows.

**(c)** Replace the third row with the third row minus four times the first row.

**(d)** The previous two steps eliminated the $x$ coefficient from the second and third equations. Take a similar step to eliminate the $x$ coefficient from the fourth equation.

**(e)** You have now used the first row of the matrix to make the first column 0 in all the *other* rows. In a similar way, use the (new) second row of the matrix to make the second column zero in the third and fourth rows.

**(f)** Use the third row to make the third column zero in the fourth row.

**(g)** Explain how you can determine that your matrix is now in row echelon form.

**(h)** Based on your result, identify the rank of the original matrix. Identify the original equations as inconsistent, linearly dependent, or having a unique solution. If they are linearly dependent, specify how many variables you could solve for. If there is a unique solution, find it.

**7.86** *[This problem depends on Problem 7.85.]* For these two variations of Problem 7.85, determine if the equations are linearly independent (and find the rank of the matrix), inconsistent, or have a unique solution (and find it).

**(a)** Solve the system of equations from Problem 7.85 changing the constant term on the right of the first equation from 2 to 1.

**(b)** Solve the system of equations from Problem 7.85 again, with the constant term in the first equation back to its original 2, but this time changing the coefficient of $t$ in the fourth equation from 11 to 24.

For Problems 7.87–7.98 row reduce the augmented matrix for the given set of equations. Give the rank of the augmented matrix and say whether the equations are inconsistent, linearly dependent, or neither. If they are linearly dependent, how many variables could you solve for? Give the unique solution to the equations if there is one.

**7.87** $x + y = 3,\ x - y = 2$

**7.88** $x + 2y + 3z = 0,\ x - 2y - 3z = 0,\ 2x + 4y = 0$

**7.89** $x + 2y + 3z = 0,\ x - 2y - 3z = 0,\ 3x - 2y - 3z = 0$

**7.90** $x + y + z = 2,\ x - 2y + 3z = -1,\ 3x - 3y + 4z = 3$

**7.91** $x - 2y + z = 3,\ x + z = 4,\ x + 4y + z = 2$

**7.92** $x + 2y - z = 2,\ -2x - 4y + 2z = -4,\ 3x + 6y - 3z = 6$

**7.93** $2x - y + 3z + t = -2,\ 6x + y + 4z - 2t = -30,\ 4x + 6y + 3t = 36,\ -10x + y - 2z - 4t = 2$

**7.94** $x - z + 2t = -1,\ x - y + z - t = 3,\ 3x - 2z = 4,\ 2x - 3y - z + t = 1$

**7.95** $x + y + 6z + 7t = 21,\ 4x - 5y + 8z + t = -22,\ 9x + 8z + t = -17,\ x + y + 6z + 7t = 21$

**7.96** $4a - 7c + 4d = -11,\ 4a + 7b - 6c + 6d = 22,\ 4a + 7b + 9c - 4d = 17,\ 8a + 7b + 2c = 9$

**7.97** $-a - 3b - c + 6d = 2,\ 9a + 6b - 3c - 8d = -22,\ 2a + 6b - 3c + 5d = 25,\ 7a - 5d = -31$

**7.98** $a - b + c - d = 1,\ 2a - b - c + 2d = -2,\ 3a - 2b + d = -1,\ 4a - 3b + c = 0$

[5] To preserve his anonymity we will refer to him only as "Dad."

**7.99** *True story*: A chemical engineering professor we know[5] was once creating a homework problem involving a combustion reactor in which methane ($CH_4$) and ethane ($C_2H_6$) react with oxygen ($O_2$) to form carbon monoxide ($CO$), carbon dioxide ($CO_2$), and water ($H_2O$).

$$CH_4 + \frac{3}{2}O_2 \rightarrow CO + 2H_2O$$
$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$
$$C_2H_6 + \frac{5}{2}O_2 \rightarrow 2CO + 3H_2O$$
$$C_2H_6 + \frac{7}{2}O_2 \rightarrow 2CO_2 + 3H_2O$$

This reaction involves six molecular species. The initial concentrations of these species would be given as part of the problem. Of the six final concentrations, some would be given as "measured values" and the students would calculate the rest. According to the method he was using, the number of measured values he would have to give equals the number of species minus the number of reactions. In this example he would specify two final concentrations (6–4) and the students would calculate the other four. But his calculations did not lead to consistent answers. In this problem you will see why, and how many final concentrations he needed to specify in order to make his problem work.

**(a)** Let $A$ represent the final concentration of $CH_4$, $B$ the final concentration of $C_2H_6$, and $C$, $D$, $E$, and $F$ the final concentrations of $O_2$, $CO$, $CO_2$, and $H_2O$, respectively. Write an algebraic equation to represent each chemical reaction by replacing each chemical species with the symbol for its concentration and the arrow with an equal sign. For example, the first reaction equation would be $A + (3/2)C = D + 2F$.

**(b)** Consider $A$ and $B$ to be "constants" and the remaining concentrations as variables. (We could just as well have chosen any other two variables.) Rearrange your equations into standard form, with the variables on the left and the constants on the right. Simplify the equations so that all coefficients are integers.

**(c)** Suppose you determined, using either a determinant or row reduction, that your four equations have a unique solution. What would that tell you about the reaction? That is, what would you measure and what could you then calculate?

**(d)** Write an augmented matrix for your four equations and reduce it to row echelon form.

**(e)** Mathematically—forget about chemistry for a moment—what does your final matrix tell you about the equations you started with?

**(f)** Now let's return to our chemical engineer. He will have to measure the concentrations of some of his species, and then he will be able to calculate the rest. How many does he need to measure, and how many can he then calculate?

**(g)** We found in this case that the rows of the augmented matrix were linearly dependent, and that in turn told us something about the reaction. What different lesson would we draw if the rows of the augmented matrix were inconsistent?

**7.100** *[This problem depends on Problem 7.99.]* The "water gas shift" process can be described in terms of the following three reactions.

$$H_2O + CO \rightarrow CO_2 + H_2$$
$$H_2O + H \rightarrow H_2 + OH$$
$$OH + CO \rightarrow CO_2 + H$$

Assuming the initial concentrations of all six species are given, how many of the final concentrations would you need to measure before the rest would be determined?

**7.101** *[This problem depends on Problem 7.99.]* An experiment involves the following five reactions. The initial concentrations of all six species are known. You are going to measure as many final concentrations as you have to, and then calculate the remaining concentrations based on those measurements. How many concentrations do you have to measure, and how many can you then calculate?

$$2N_2O_5 \rightarrow 2N_2O_4 + O_2$$
$$N_2O_5 \rightarrow NO_2 + NO_3$$
$$NO_3 \rightarrow O_2 + NO$$
$$NO + N_2O_5 \rightarrow 3NO_2$$
$$2NO_2 \rightarrow N_2O_4$$

You may have a computer do the row reduction; your job is to set up the matrix and then interpret the results.