3.7 Additional Problems

For Problems 3.129–3.137 find the real part, the imaginary part, the modulus, and the phase of the given number.

3.129 $p/i + (p+i)/(5-i)$ where p is a real number

3.130 $e^{3i}/(2+i)$

3.131 $(1-e^{2i})^{40}$

3.132 $3\cos(\pi/6) - 3i\sin(\pi/6)$

3.133 $3\sin(\pi/6) - 3i\cos(\pi/6)$

3.134 $3\cos(\pi/6) - 2i\sin(\pi/6)$

3.135 $\sqrt{e^{pi}}$ where p is a real number

3.136 i^i

3.137 $\ln(2-2i)$

3.138 Let $w = z^i$.

(a) Write an expression for $\ln w$ as a function of $|z|$ and ϕ_z. Simplify as much as possible.

(b) Exponentiate your expression for $\ln w$ to get an expression for w as a function of $|z|$ and ϕ_z.

(c) Write $|w|$ and ϕ_w as functions of $|z|$ and ϕ_z, simplifying your answers as much as possible.

3.139 **(a)** Write $\sin x$ in terms of complex exponentials. *Hint*: You can start by writing e^{ix} and e^{-ix} in terms of trig functions.

(b) Write $\cos x$ in terms of complex exponentials.

(c) Take the derivative of your complex exponential formula for $\sin x$. Explain why your result was predictable.

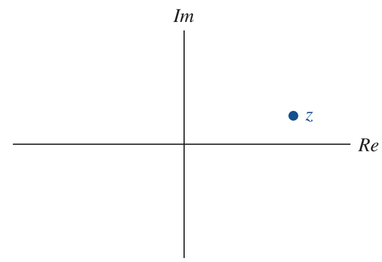
(d) The "hyperbolic trig functions" are defined as $\sinh x = (1/2)(e^x - e^{-x})$, $\cosh x = (1/2)(e^x + e^{-x})$. Write expressions for $\sinh(ix)$ and $\cosh(ix)$ in terms of real trig functions.

(e) Write expressions for $\sin(ix)$ and $\cos(ix)$ in terms of real hyperbolic trig functions.

3.140 Find the two solutions to the quadratic equation $4x^2 + 8x + 5 = 0$.

3.141 Solve for the real quantities r and s: $s^2 - 2s^2 i - 2r + ri = 9 - 18i$.

3.142 The figure shows a point z on the complex plane. Copy the figure. Then add the following points, and label them.



(a) z^* (the complex conjugate of z)

(b) $3z$

(c) $z \times i$

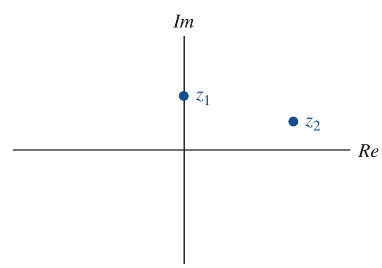
(d) $z \times 2e^{(\pi/6)i}$

3.143 Copy the drawing from Problem 3.142.

(a) Draw all the points in the complex plane that have the same magnitude as the given point z. What shape does your drawing represent?

(b) Draw all the points in the complex plane that have the same phase as the given point z. What shape does your drawing represent?

3.144 The drawing shows two complex numbers z_1 and z_2 on the complex plane. $|z_1| = 1$ and $|z_2|$ is unspecified (but greater than 1).



(a) Is z_1 a real number (such as 5), an imaginary number (such as $3i$), or neither (a number $a + bi$ where $a \neq 0$ and $b \neq 0$)?

(b) Is z_2 a real number, an imaginary number, or neither?

(c) Answer in words: if I take any complex number in the world (not necessarily z_2) and multiply it by z_1, what happens to its magnitude? What happens to its angle? Visually, what happens to the point?

(d) Copy the drawing into your homework. Then add to the drawing the labeled point $z_1 - z_2$.

(e) Add to your drawing the labeled point $z_1 z_2$.

3.145 If you specify a complex number with a free (real) variable, the result is generally a curve in the complex plane. For instance, the number $z = 3 + ki$ for $-2 \le k \le 2$ describes a vertical line segment four units long. Draw the curve described by each of the following functions. (Remember that in all cases k is a real number!)

(a) $z = k - i$ for $0 \le k \le 5$

(b) $z = 3k + ki$ for $0 \le k \le 5$

(c) $z = k + k^2 i$ for $-3 \le k \le 3$

(d) $z = 2e^{ik}$ for $0 \le k \le \pi/2$

(e) $z = ke^{i\pi/6}$ for $0 \le k \le 2$

(f) $z = ke^{ik}$ for $0 \le k \le \pi/2$

3.146 Prove that any complex number z and its complex conjugate z^* are the same distance from the origin on the complex plane.

3.147 You solve for the motion of a mass on a damped spring and one of the solutions you get is $x(t) = 3e^{(2-3i)t}$. How can you know without doing any more calculations that you made a mistake in deriving this solution?

3.148 You find the following particular solution describing the motion of an oscillator: $x(t) = (3 - 2i)e^{(-4+5i)t}$.

(a) What is the initial amplitude of the oscillations?

(b) What is the period?

(c) How long will it take before the amplitude drops to 1% of its initial value?

(d) Write a complex function $y(t)$ for an oscillator that is behaving exactly like the first one, but with oscillations $\pi/2$ behind it in phase.

3.149 A light beam moving in the z-direction is often represented by specifying the vector form of the electric field that forms part of the light beam.

$$\vec{E} = e^{i(kz-\omega t)}\left(E_{0x}e^{i\phi_x}\hat{i} + E_{0y}e^{i\phi_y}\hat{j}\right)$$

(Light waves are "transverse," meaning a light wave moving in the z-direction has no electric field component in the z-direction.) The electric field is actually the real or imaginary part of this expression, but for most purposes people work directly with the complex form.

(a) What is the wavelength of the light beam?

(b) What is the period of the light beam?

(c) Suppose $k = 2\text{m}^{-1}$, $\omega = 500\text{s}^{-1}$, $E_{0x} = 100$ N/C, $E_{0y} = 300$ N/C, $\phi_x = \pi$, $\phi_y = \pi/6$. How long will it be from the time the x-component peaks to the time the y-component peaks?

In Problems 3.150–3.153, find the general real-valued solution to the given differential equation.

3.150 $d^2x/dt^2 - 2(dx/dt) + 10x = 0$

3.151 $5(d^2y/dx^2) + 4(dy/dx) + 8y = 0$

3.152 $2(d^2r/dt^2) - 10(dr/dt) + 13r = 0$

3.153 $4(d^2x/dt^2) + 10(dx/dt) + 7x = 0$

3.154 The differential equation $d^2f/dt^2 + df/dt = \sin f$ is non-linear and has no simple solution.

(a) Assume $f(t)$ stays close to π. Replace the right side of the differential equation with the linear terms of its Taylor series about $f = \pi$. Your answer should be a linear differential equation for $f(t)$.

(b) Solve the resulting approximate differential equation.

(c) This approximation will only work at late times if $f(t)$ stays close to π. Based on your solution, is it valid to assume that if f starts out close to π with a small derivative, it will stay close to π at late times? Explain.

3.155 Evaluate $\int e^x \sin(ix)dx$. *Hint*: Begin by expressing $\sin(ix)$ as a sum of exponentials. See Section 3.4, Problem 3.90 if you're stuck.

3.156 An oscillating system obeys the differential equation $x''(t) + ax'(t) + x = 0$. Solve this equation for $a = 1$ and $a = 3$. In each case find a real-valued, general solution and sketch the basic shape of that solution $x(t)$.

3.157 An oscillator obeys the equation $x''(t) + ax'(t) + 2x = 0$.

(a) Find the general solution to this equation. Your answer will include the unknown constant a.

(b) For what values of a will the system oscillate?

(c) What will the system do if a is not in the range you just found?

3.158 A damped pendulum obeys the differential equation $\theta''(t) + 3\theta'(t) + 3\sin(\theta) = 0$. Find a linear differential equation that approximates this well for small amplitudes and find the general solution to that linear differential equation. Will the pendulum oscillate or simply decay?

3.159 Find the general real-valued solution to the equation $x''(t) + 2x'(t) + 5x = 3\sin(2t)$. Express your final answer as a real-valued function with two arbitrary constants. *Hint*: You will need to find a complementary solution and a particular solution.