CHAPTER 2

Taylor Series and Series Convergence (Online)

2.8 Asymptotic Expansions

In introductory calculus classes the statement "this series diverges" is generally taken to mean "this series is useless." But with *asymptotic expansion* we can sometimes use divergent series to approximate functions.

2.8.1 Explanation: Asymptotic Expansions

On Page 70 we presented the following Taylor series.

$$\frac{1}{1-x} = -\frac{1}{2} + \frac{1}{4}(x-3) - \frac{1}{8}(x-3)^2 + \frac{1}{16}(x-3)^3 + \ldots \qquad (1 < x < 5) \tag{2.8.1}$$

Equation 2.8.1 claims that the function on the left and the infinite series on the right are "equal"—that is, the partial sums $S_n(x)$ are roughly equal to the function values $f(x)$. This approximation works best if n is very large and x is very close to 3. We can therefore make this claim more specific in two ways.

1. Hold x constant and increase n. For instance if $x = 3.1$ then $1/(1-3.1) \approx -0.47619$.

$$\begin{aligned} S_1(3.1) &= -1/2 && = -0.5 \\ S_2(3.1) &= -1/2 + (0.1)/4 && = -0.475 \\ S_3(3.1) &= -1/2 + (0.1)/4 - (0.1)^2/8 && = -0.47625 \end{aligned}$$

   First claim: **As $\mathbf{n \to \infty}$, $\mathbf{S_n(3.1) \to 1/(1-3.1)}$**. This claim holds, not only for $x = 3.1$, but for any x-value between 1 and 5.

2. Hold n constant and let x approach 3. For instance, S_2 is the linear approximation $-1/2 + (x-3)/4$.

$$\begin{aligned} f(4) &= -0.333 & S_2(4) &= -0.25 \\ f(3.1) &= -0.47619 & S_2(3.1) &= -0.475 \\ f(3.01) &= -0.497512 & S_2(3.01) &= -0.4975 \end{aligned}$$

   Second claim: **The closer x gets to 3, the closer $\mathbf{S_2(x)}$ comes to $\mathbf{1/(1-x)}$.** This claim holds, not only for $S_2(x)$, but for any partial sum in the series.

Take a moment to convince yourself that these two claims define our expectation for any Taylor series, and that both are true for Equation 2.8.1.

For a divergent series the first claim above cannot possibly be true. (In the limit as $n \to \infty$ such a series does not approach anything.) In some such cases, however, the second claim still holds true; each partial sum makes a better and better approximation for a given function as x gets closer to some designated value. Since most of our uses for power series involve using finite partial sums as approximations, that claim is enough to make a power series useful even if it will ultimately diverge.

The Asymptotic Expansion for the Complementary Error Function

As an example, consider the "complementary error function" erfc x.

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt$$

This function comes up in probability and statistics, and we discuss it in Chapter 11. For our purpose here you only need to know three things about the complementary error function: it is useful in many real-world situations, it is defined for all x-values, and it can be difficult to calculate. It is therefore desirable to approximate its values with a series.

$$\text{erfc } x \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!(2x)^{2n}} \tag{2.8.2}$$

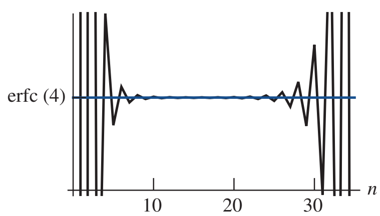

FIGURE 2.13 Partial sums of the asymptotic series for erfc x, evaluated at $x = 4$.

We will discuss below where that series comes from, but first let's see what it does. If you choose one particular x-value and start accumulating terms you will find that the partial sums approach the desired value for a while and then move away from it. For instance Figure 2.13 shows partial sums of this series evaluated at $x = 4$. By the 10th partial sum the series gives a very good approximation of erfc 4, but some time after the 20th it begins to move away.

You can see that this series does not ultimately converge to erfc 4. In fact Equation 2.8.2 diverges for any x-value you plug into it! (See Problem 2.183.) But you can also see that this series does approximate erfc 4 well if you add up the right number of terms. (The optimal result often comes from stopping after the smallest term—not a surprising result if you look at the graph.)

Asymptotic Expansions

The above example shows that a divergent series can still be useful for estimating a function value. But there are two other important points we need to make—about this example in particular, and asymptotic expansions in general.

Suppose you choose a particular partial sum—the best one in our example above is the 15th. Now instead of an infinite series you have a *finite* series that can be used to accurately approximate erfc 4. But that particular finite series makes an even better approximation for erfc 5, and it's spectacular for erfc 100. Every asymptotic expansion is built around a particular value or (as in this case) around infinity, and the approximation works better as you approach that value. Above we made two claims about Equation 2.8.1; here we are saying that the *second* of these two claims also holds for Equation 2.8.2.

If that were the end of the story we could just work with the finite series and forget the infinite divergent series that we started with. In practice we often do just that. But suppose you use more terms: say, the 40th partial sum. We saw above that for $x = 4$ that makes a lousy approximation. But for sufficiently high x-values it makes a very good approximation—better in fact than S_{15}. As x gets higher we can use more and more terms of the series.

In summary: when we claim that a series $\Sigma a_n(x)$ asymptotically approaches a function $f(x)$ around $x = x_0$ we are saying the following two things. (Note that x_0 could be a finite number or, as in the example above, infinity.)

1. Any given partial sum $S_n(x)$ can make an arbitrarily accurate approximation to $f(x)$ by allowing x to approach x_0. (We say below how we are defining accuracy.)
2. As x gets closer to x_0 you can use higher partial sums before they start diverging away from the correct value. This is a consequence of the first point, but we note it separately because it is useful for understanding asymptotic series.

The definition below does not just reexpress the points we made above in more formal language; it provides a specific requirement for how the partial sums must approach the function.

Definition: Asymptotic Expansion

Let S be the series $\sum_n a_n(x)$ and let S_n be the nth partial sum of S. We say S is an asymptotic expansion of a function $f(x)$ about the point $x = x_0$ if it obeys the following limit for any fixed, positive integer n.

$$\lim_{x \to x_0} \frac{f(x) - S_n}{a_n(x)} = 0$$

For a Taylor series that converges to a function we write $f(x) = S$. For a series that is divergent but asymptotically approaches $f(x)$ in the sense defined above, we write $f(x) \sim S$.

Deriving an Asymptotic Series

Equation 2.8.2 gives an asymptotic series for the complementary error function. Below we start the process of deriving that formula. In the problems you will continue this process and show why it gives us an asymptotic series.

Our strategy is use integration by parts to tackle the integral that defines the complementary error function.

$$\text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

$$u = \frac{1}{t} \qquad dv = te^{-t^2}\, dt$$
$$du = -\frac{1}{t^2}\, dt \qquad v = -\frac{1}{2} e^{-t^2}$$

$$\frac{2}{\sqrt{\pi}} \left[uv - \int v\, du \right] = \frac{2}{\sqrt{\pi}} \left[-\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{1}{t^2} e^{-t^2}\, dt \right]_x^\infty$$

The uv term (before the integral) vanishes at $t = \infty$ so plugging in the limits of integration gets us here.

$$\text{erfc } x = \frac{e^{-x^2}}{\sqrt{\pi}\, x} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^2} e^{-t^2}\, dt \qquad (2.8.3)$$

Equation 2.8.3 is not an approximation; it is an exact rewriting of the complementary error function. The term before the integral represents the first term in the asymptotic series expansion for erfc x. The integral itself represents the remainder—that is, the difference between the actual function and the one-term series approximation.

In Problem 2.189 you will take this process to the next step, finding subsequent terms in the series and the integrals that represents their remainders. In Problem 2.190 you will use the remainder terms to explain the behavior of this asymptotic series.

2.8.2 Problems: Asymptotic Expansions

2.183 Prove that the asymptotic series for erfc x given in Equation 2.8.2 is divergent for any fixed value of x.

2.184 Equation 2.8.2 gives a series expansion for erfc x. Suppose you wish to use this series to approximate erfc 3.

(a) Evaluate terms of this series expansion at $x = 3$. You should see that the terms decrease in magnitude for a while and then increase. What term has the smallest magnitude?

(b) In Part (a) you found the n-value that minimizes the magnitude of the terms in this series. Make a graph of the *partial sums* of this series as n goes from 0 up to twice that value. Include on your graph a label showing the correct value of erfc 3. Describe the behavior you see.

(c) How many terms of this series do you need to obtain an approximation accurate to within 0.1%?

(d) Repeat Parts (a)–(c) for erfc 5. How is the behavior the same, and how is it different?

2.185 The "exponential integral" is defined as Ei $x = -\int_{-x}^{\infty}(e^{-t}/t)dt$. The following series converges to Ei x for all $x \neq 0$: Ei $x = \gamma + \ln|x| + \sum_{n=1}^{\infty} x^n/(nn!)$. Here γ is the "Euler–Mascheroni constant," roughly equal to 0.5772. You can also represent Ei x with the following asymptotic expansion, valid in the limit $x \to -\infty$.

$$\text{Ei } x \sim \frac{e^x}{x}\sum_{n=0}^{\infty}\frac{n!}{x^n}$$

You'll use both of these series to approximate Ei(−10).

(a) Prove that this asymptotic series diverges for any value of x.

(b) Calculate the value of Ei(−10) to at least 5 decimal places.

(c) Calculate the 40th partial sum of both series. Is either one a good approximation?

(d) Calculate the 30th partial sum of both series. Is either one a good approximation?

(e) Calculate the 3rd partial sum of both series. Is either one a good approximation?

(f) Plot the partial sums of both series up to $N = 50$ as a function of N (the maximum value of n used in the partial sum). Show on your plots the correct value of Ei(−10). Describe how each series behaves.

(g) Why is it useful to have a divergent, asymptotic expansion even though there is a convergent series that works for this function?

2.186 Suppose you use n terms of an infinite series to approximate a value X. The "remainder" is the difference between the actual value and your approximation: $R_n = |X - S_n|$.

(a) Draw a graph of R_n as a function of n for a convergent Taylor series. (Although the details vary from one convergent Taylor series to the next, the overall shape should be the same.)

(b) Draw a graph of R_n as a function of n for a divergent asymptotic series. (Same comment.)

2.187 Consider the function $f(x) = \int_x^{\infty} e^{-t^3}\,dt$.

(a) Use integration by parts to find the first two terms of an asymptotic series expansion for $f(x)$. Express the remainder as an integral.

(b) For what values of x does your two-term series best approximate $f(x)$? (Does it work best for values of x close to zero, values of x close to some other number, or values of x approaching ∞?) How can you tell?

(c) Use your two-term series to approximate $f(4)$, and compare it to the actual value of $f(4)$.

2.188 The "error function" is defined by the same integral as the complementary error function, but with different limits of integration.

$$\text{erf } x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

The behavior and use of the error function are discussed in Chapter 11.

Our purpose here is to approximate this function with two different series and compare their behavior. It can be shown that erf $x = 1-$ erfc x, so its asymptotic expansion is

$$\text{erf } x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{n!(2x)^{2n}}$$

(a) Write the Maclaurin series for e^t. (You can derive it or look it up.) From there you should be able to easily generate the Maclaurin series for e^{-t^2} and from there the error function. You should not need a computer for this part, although you can use one to check your answer when you're done.

You now have two different series that can be used to estimate erf x. In the remainder of this problem you will compare these two series using a computer.

(b) Use the $n = 3$ partial sum of each series to estimate erf 1/2. Which estimate is more accurate?

(c) Use the $n = 3$ partial sum of each series to estimate erf 2. Which estimate is more accurate?

(d) On one plot, show erf x and the $n = 3$ partial sum of both series for $0 \le x \le 5$. Choose a vertical range that allows you to see when each series is and isn't a good approximation to the function, and estimate the ranges in which each one gives a good estimate.

(e) One one plot, show the partial sums of both series with $x = 2$ as a function of N, the highest n-value of the partial sums. Estimate the range of partial sums for which each series gives a good approximation to erf 2.

2.189 Equation 2.8.3 gives the first term in the series expansion of erfc x.

(a) Use integration by parts on that series to find the next term. Your final answer will be in the form erfc x = *<two terms plus an integral>*. *Hint*: You cannot integrate e^{-t^2} but you can integrate te^{-t^2}.

(b) After n integrations the series looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$. Assume the remainder term is given by the following integral with some coefficient c_n.

$$R_n = c_n \int_x^{\infty} \frac{1}{t^{2n}} e^{-t^2}\, dt$$

Perform the next integration by parts to find the term a_{n+1} and the remainder R_{n+1}. You should verify that the pattern continues because R_{n+1} looks like R_n with n replaced by $n+1$ and a different coefficient.

(c) Do one more integration by parts to find a_{n+2}. (Don't worry about the remainder integral this time.) Simplify the ratio $|a_{n+2}/a_{n+1}|$.

(d) Given a fixed value of x, for which values of n will a_{n+2}/a_{n+1} be greater than 1, and for which will it be less than 1?

(e) Using your answer to Part (d), explain why Figure 2.13 looks like it does. In particular, explain how you can use that answer to predict the value on the horizontal axis at which the terms switch from converging toward a finite value to diverging away from it.

Your results in this problem demonstrate that for any fixed x the asymptotic series for erfc x should converge toward a finite value for a while, and then start diverging above a certain value of n. Problem 2.190 will continue this argument by arguing that the finite value the series converges to is in fact erfc x.

2.190 **Exploration: The Remainder of the Asymptotic Expansion for erfc** *[This problem depends on Problem 2.189.]*

At any particular value of n, the asymptotic series for erfc x looks like $\sum_{n=0}^{\infty} a_n(x) + R_n$, where R_n is the exact difference between the series expansion and the correct value of erfc x. In this problem you're going to show that as n increases this remainder term decreases for a while, indicating that the series is getting closer to erfc x, but that it starts to increase past a certain value of n. In fact you'll show that this is approximately the same value of n at which the terms a_n go from decreasing to increasing.

In Problem 2.189 you showed that if $R_n = c_n \int_x^{\infty} t^{-2n} e^{-t^2} dt$ then $R_{n+1} = -c_n[(2n+1)/2] \int_x^{\infty} t^{-(2n+2)} e^{-t^2} dt$. The coefficient in front of the R_{n+1} integral is bigger than the one in front of the R_n integral by a factor of $(2n+1)/2$. At the same time, however, the integrand decreased by a factor of t^2, and it's harder to say what effect that has on the entire integral. The key to figuring that out is to notice that the decaying exponential causes the integral to be dominated by values of t very close to x.

(a) Numerically calculate $\int_x^\infty t^{-2}e^{-t^2}\,dt$ and $\int_{x+.1}^\infty t^{-2}e^{-t^2}\,dt$ for values of x ranging from 2 to 20. Show that for sufficiently large x over 90% of the entire integral comes from $x < t < x + .1$. Estimate the lowest value of x for which is true.

(b) Numerically calculate the ratio of $\int_x^\infty t^{-2}e^{-t^2}\,dt$ to $\int_x^\infty t^{-4}e^{-t^2}\,dt$ for values of x ranging from 2 to 20. On the same plot, plot x^2.

(c) What does dividing the integrand by t^2 do to the value of the integral? Explain how your answer follows from the plots you made.

(d) Putting together your answers so far, write a simple approximation for R_{n+1}/R_n, valid for large x.

(e) Using your answer to Part (d), estimate the value of n at which the remainder term switches from decreasing to increasing as you increase n. Check your answer by verifying that it correctly predicts the appearance of Figure 2.13.

(f) Explain how the results you've derived in this problem lead to the two properties that we said define an asymptotic series. *Hint*: Remember that this integral represents the difference between the original function and the nth term of the asymptotic series!