13.11 Additional Problems

13.147 **(a)** Write the real and imaginary parts of $\ln z$ as functions $u(x, y)$ and $v(x, y)$ (where $z = x + iy$).

(b) Show that $\ln z$ is analytic by showing that u and v satisfy the Cauchy-Riemann equations. It helps to know that the derivative of $\tan^{-1} x$ is $1/(1 + x^2)$.

13.148 The function $f(z) = (\cos z)/(z - \pi/4)^2$ has a singularity at $z = \pi/4$.

(a) Find the residue at this singularity using the formula given in Section 13.4.

(b) Find the Laurent series for $f(z)$ about the point $z = \pi/4$. *Hint*: Start by writing $\cos z = \cos[(z - \pi/4) + \pi/4]$ and use the cosine addition rule.

(c) Is the singularity at $z = \pi/4$ removable, essential, or a pole? If it's a pole, what order is it? If it's a pole or an essential singularity, what is the residue? Explain how you know the answers based on the Laurent series.

(d) Find the contour integral of $f(z)$ around a unit circle centered on the origin.

13.149 The multi valued function $f(z) = \sqrt{1 + z^2}$ can be made single-valued with two branch cuts. The starting point is to rewrite $f(z)$ as $\sqrt{z - i}\sqrt{z + i}$.

(a) First consider $\sqrt{z - i}$. Where is the branch cut for this function (using the principal phase of $z - i$)?

(b) Where is the branch cut for $\sqrt{z + i}$?

(c) Draw a z-plane with a small circle around $z = i$. Sketch the curve this circle maps to on the f-plane. What happens to $f(z)$ as the z circle crosses a branch cut?

13.150 *[This problem depends on Problem 13.149.]* In this problem you will make plots of mappings of the function $f(z) = \sqrt{1 + z^2}$ using the branch that has two branch cuts, as discussed in Problem 13.149.

(a) Have a computer make a z-plane with the circle $|z| = 2$ and an f-plane with the curves this circle maps to.

(b) What are the values of f just before and after crossing each branch cut in this mapping?

(c) Repeat Parts (a)–(b) for the circle $|z - i| = 1$. The curve will only cross one branch cut this time.

13.151 In this problem you will define $\sqrt{z - i}$ and $\sqrt{z + i}$ with branch cuts extending vertically downward from their poles. For example, $\sqrt{z - i}$ will have a branch cut going from $z = i$ to $z \to -i\infty$. Your goal is to find the branch cut(s) for the function $f(z) = \sqrt{1 + z^2} = \sqrt{(z - i)(z + i)}$.

(a) Start by drawing a complex plane. Throughout the problem you are going to mark points z and $f(z)$ on it.

(b) For $z_1 = 2i$ find the phases of $\sqrt{z + i}$, $\sqrt{z - i}$, and $f(z)$. If any of these phases is ambiguous due to the branch cut, find instead the phases of $z_1 + \epsilon$ and $z_1 - \epsilon$ where ϵ is a small positive real number. Mark and label the points z_1 and $f(z_1)$ on your plot. (If it lands at different places depending on whether you add or subtract ϵ then mark them both.)

(c) Repeat Part (b) for $z_2 = i/2$.

(d) Repeat Part (b) for $z_3 = -i/2$.

(e) Repeat Part (b) for $z_4 = -2i$.

(f) A branch cut for a function $f(z)$ is a line where the position of $f(z)$ is different depending on which side you approach that line from. Looking at the plot you made, where is the branch cut for this $f(z)$?

13.152 Let $f(z) = 1/(1 + z)$.

(a) Let $z = x + iy$ and $f = u + iv$. Find $u(x, y)$ and $v(x, y)$.

(b) What is the (constant) value of $u(x, y)$ on the unit circle $x^2 + y^2 = 1$ (not counting the pole at $(-1, 0)$)?

(c) What is the value of $u(x, y)$ in the limit $|z| \to \infty$?

(d) Write a boundary value Laplace's equation problem that $u(x, y)$ is the solution to.

13.153 Evaluate the following (real-valued) integral.

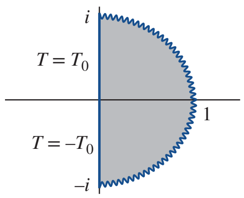
$$R = \int_0^\infty \frac{1}{x^k(x - x_0)}\,dx \text{ where } 0 < k < 1,\ x_0 > 0$$

Hint: Your final answer will probably look complex. You should be able to simplify it by replacing x_0 with $|x_0|e^{i\pi}$, but don't do this until you get to the final answer.

13.154 Let $f(z) = 1/\sin z$.

- **(a)** What is the order of the pole at $z = 0$?
- **(b)** Calculate the residue of f at $z = 0$ using the formula on Page 730.
- **(c)** Find the first three terms of the Laurent series for $f(z)$ about $z = 0$.
- **(d)** Use your Laurent series to find the residue at $z = 0$ and confirm that you get the same answer you did in Part (b).

13.155 The picture below shows half the unit disk in the uv-plane. The straight boundary is split in two, with a temperature $T = -T_0$ on the bottom half and $T = T_0$ on the top. The curved boundary is insulating, meaning the derivative of T normal to that boundary is zero.



- **(a)** The steady-state temperature in the region obeys Laplace's equation. Find that steady-state temperature $T(u, v)$.
- **(b)** The transformation $z(f) = e^f$ maps this half-disk to a different region. Make a rough sketch of that region. Mark on your mapped region where the $T = -T_0$ boundary, the $T = T_0$ boundary, and the insulating boundary from the original region mapped to.
- **(c)** Solve Laplace's equation in the mapped region subject to the boundary conditions you just added to your sketch.

13.156 An infinite wire is at potential $V = V_L$. A circle of radius R is tangent to that wire, and is held at $V = V_C$. Define a useful set of axes for this problem, find a Möbius transformation that maps the circle and line to two parallel lines, and use that mapping to solve Laplace's equation for the potential in the region bounded by the circle and the line. (That region is shaped like a half plane with a disk cut out of it.)

13.157 Sometimes the best way to solve a difficult problem is to find an easier problem that's similar enough that it gives a good approximate answer. In this problem you'll solve for the steady-state temperature in a rectangle stretching from the origin to the point $(\pi/2, 5)$, with the left edge fixed at $T = 1$ and the bottom and right edges fixed at $T = 0$. The top edge is insulating, which means the derivative of T normal to that boundary is zero.[13]

- **(a)** Write $f(z) = \sin(x + iy)$ in the form $f = u(x, y) + iv(x, y)$. This is a bit of a mess, but you can do it by using Euler's formula $e^{iz} = \cos z + i \sin z$ and the related formula $e^{-iz} = \cos z - i \sin z$.
- **(b)** Use the function $\sin z$ to map the left, bottom, and right boundaries of the rectangle to the uv-plane. You should find that they all map to line segments.
- **(c)** Find the curve that the top boundary maps to. You should be able to simplify its formula to the form $f = p \sin a + iq \cos a$, where p and q are real constants and a is a parameter that varies along the curve. Find the numbers p and q and specify the range of values for a.

If p equaled q this curve would be an arc (part of a circle). If that were the case it would be relatively easy to solve Laplace's equation to find $T(u, v)$ in the mapped region. Since $p \neq q$ it's actually an ellipse, and it's not easy to find $T(u, v)$. However, since $p \approx q$, you can get a good approximation by pretending that it's an arc.

- **(a)** Assuming the curved boundary is an arc, write the solution $T(u, v)$ that solves Laplace's equation and meets all the boundary conditions in the mapped region.
- **(b)** Use the functions u and v to map your solution $T(u, v)$ back to the xy-plane. You should get an answer in the form $T(x, y)$.
- **(c)** Your answer should correctly solve Laplace's equation and meet the left, bottom, and right boundary conditions. It will *not* meet the boundary condition at the top, which is $\partial T/\partial y = 0$. Calculate $\partial T/\partial y$ at the point $(\pi/4, 5)$ and verify that it is not zero, but is very close. As a point of comparison calculate $\partial T/\partial y$ at the point $(\pi/4, 1)$.

13.158 **Exploration: The Uniqueness Theorem for Laplace's Equation**
Throughout this chapter you've been solving the Dirichlet problem for Laplace's equation: you are given the value of a

[13] If you worked Section 13.9 Problem 13.131 you may recognize this as a finite version of the same problem. As it often does, making the domain of the problem finite makes it harder.

function $V(x, y)$ on the boundaries of a region and asked to find a harmonic function $V(x, y)$ that meets those boundary conditions. In this problem you will prove that each such problem has a unique solution. First you'll prove another remarkable property of harmonic functions: the average value of a harmonic function $V(x, y)$ on a circle centered on the point (x_0, y_0) equals $V(x_0, y_0)$.

(a) Consider an analytic function $g(z)$. In Cauchy's integral formula, Equation 13.4.1, let the contour be a circle of radius R around centered on z_0. Such a circle can be described as $z = z_0 + Re^{i\phi}$. By rewriting the integral in that form, show that the average value of $g(z)$ on this circle equals $g(z_0)$.

(b) Use your result from Part (a) to argue that the average value of a harmonic function $V(x, y)$ on a circle of radius R must equal the value of V at the center of that circle.

(c) Use your result from Part (b) to argue that a function that is harmonic everywhere in a closed region cannot take on a global maximum or minimum value anywhere in the interior of that region unless it also takes on that value on the boundary.

(d) Suppose you knew the values of a function $V(x, y)$ everywhere on the boundary of a region and you found two harmonic solutions V_1 and V_2 satisfying those boundary conditions. Prove that $V_1 - V_2 = 0$. That completes the proof that the solution to a boundary-value Laplace's equation problem must be unique.

A corollary of this result is Liouville's Theorem, which says that an entire function whose magnitude is bounded must be constant. We will leave it to you to think about how that follows from what you derived in this problem.