13.10 Special Application: Fluid Flow

We have seen how analytic functions can be used to solve Laplace's equation under a variety of boundary conditions—first in simple regions (Section 13.3) and then, using conformal mapping, in more complicated regions (Section 13.9). Our applications for these techniques have all been steady-state temperature and electrostatic potential problems. In this section we apply those same techniques to the slightly more complicated problem of fluid flow.

Velocity Fields and Stream Functions

Figure 13.18 shows a rock in a stream. Our goal is to mathematically model the flow of the water around this rock. That is, we want to find the water's velocity field $\vec{v}(x, y)$.

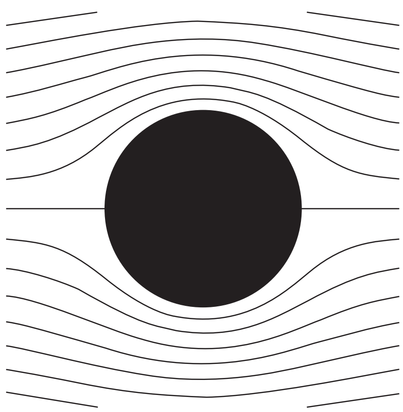


FIGURE 13.18 Flow of water around a circular obstacle.

If we neglect viscosity then the flow can be conceptually divided into thin curves of flowing liquid that exert no forces on each other. These curves, called streamlines, are shown in Figure 13.18. We could replace all the water below one of those streamlines with a solid boundary and the flow above that streamline would be unaffected.

This particular problem has two boundary conditions. The first is the rock. Because no water flows into or out of the rock, one streamline must lie directly along the top curve of the rock, and another streamline along the bottom curve. The second boundary condition is "at infinity"—far away the rock is irrelevant, so the streamlines are evenly spaced horizontal lines.

The technique we are going to present here works for this problem and many others like it, but let's start by laying out its limitations. We're going to assume throughout this section that the velocity field is divergenceless ($\vec{\nabla} \cdot \vec{v} = 0$) and irrotational ($\vec{\nabla} \times \vec{v} = \vec{0}$). We will also restrict ourselves to flow in two dimensions. While real fluid flow occurs in 3D, 2D flow is a good model for systems ranging from shallow streams to wind across an airplane wing. (Movement sideways to the airplane is not generally too significant.)

The Stream Function

Instead of directly finding the velocity function we will spend most of our efforts finding a scalar field called the "stream function" $\psi(x, y)$. We can find such a function, and then find $\vec{v}$ from it, provided $\vec{\nabla} \cdot \vec{v} = 0$ (one of the assumptions we mentioned above). We define this new function by its relationship to the vector we are looking for.

- If you have ψ and you want $\vec{v}$ you take derivatives.

$$\vec{v}(x, y) = \frac{\partial \psi}{\partial y}\hat{\imath} - \frac{\partial \psi}{\partial x}\hat{\jmath} \tag{13.10.1}$$

  This equation may remind you of how we differentiate the potential V to find the electric field $\vec{E}$, although you should certainly note the differences as well as the similarities. You will see below that the way we integrate $\vec{v}$ to find ψ is reminiscent of the way we get from $\vec{E}$ to V, but again different in important ways. We do want to draw your attention to one important similarity between the two systems: what matters is the *change* in ψ

from one point to another, not its actual value, so you can choose any point you like to set $\psi = 0$.

- If you have $\vec{v}$ and you want ψ you begin by choosing an arbitrary point (x_0, y_0) as the place where $\psi = 0$. Then draw a curve C connecting this point to a second point (x, y). At this second point ψ is the flux of $\vec{v}$ through that curve—in other words the number of streamlines passing through the curve.

$$\psi(x, y) = \int_C \left(\vec{v} \cdot \hat{n}\right) ds \qquad (13.10.2)$$

  You will show in Problem 13.141 that this flux is the same along any curve between (x_0, y_0) and (x, y). (This definition leaves the sign of ψ ambiguous since the flux through an open contour requires a decision about which direction is positive. When necessary this ambiguity can be removed using Equation 13.10.1.)
- Visually, the *contour lines* of ψ (the curves along which ψ is constant) are the *streamlines* of $\vec{v}$.

These rules are three different ways of expressing the same relationship between ψ and $\vec{v}$, but not obviously so. As one way to begin to see the connection, Figure 13.19 shows a streamline between two points and a curve C that lies directly along the streamline. Because C lies directly along a streamline, there is no flux of the stream through the curve. This means that $\psi(x_1, y_1)$ and $\psi(x_2, y_2)$ are the same as each other. So you can see why the streamlines of $\vec{v}$ become the curves along which ψ is constant.

In Problem 13.142 you will show how Equation 13.10.2 implies Equation 13.10.1.

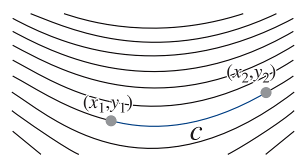


FIGURE 13.19 A curve lying directly along a streamline.

EXAMPLE The Stream Function and the Velocity Field

Question: For water flowing uniformly in the horizontal direction (such as the water far away from the rock in Figure 13.18) the streamlines are evenly spaced horizontal lines. What are $\vec{v}$ and ψ in that situation?

Answer:

In the picture $(0, 0)$ is the point we have arbitrarily chosen to represent $\psi = 0$ and (x, y) is the point where we want to find ψ. We have drawn a curve C between them. The flux through this curve—the amount of water that passes through it—is directly proportional to $y - y_0$, independent of the x-values of the two points. From Equation 13.10.2, then, $\psi(x, y) = ky$ fits this stream. We can use Equation 13.10.1 to go back the other way.

$$\psi(x, y) = ky \quad \rightarrow \quad \vec{v} = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j} = k\hat{i}$$

This correctly predicts a uniform horizontal flow. Note that Figure 13.20, which we drew to represent the streamlines of the flow, is also a picture of the contour lines of $\psi(x, y) = ky$: evenly spaced horizontal lines.

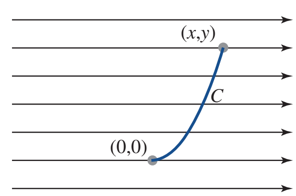


FIGURE 13.20

As a final note on this example, the function $\psi(x, y) = ky + A$ works perfectly for any constant A. $\Delta\psi$ between any two points is still proportional to $y - y_0$ (the flux), and $\partial\psi/\partial y$ is unchanged. Choosing a constant A for this solution is equivalent to choosing an arbitrary streamline on which to set $\psi = 0$.

Question: How does all that change for $\psi = ky^2$?

Answer:
The contour lines of ψ are once again horizontal, but this time they are not evenly spaced. An identical curve at a higher y-value would have more streamlines passing through it, i.e. a higher flux. (Figure 13.21.)

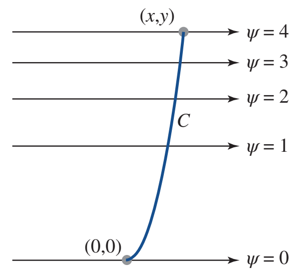


FIGURE 13.21

$$\psi(x, y) = ky^2 \quad \rightarrow \quad \vec{v} = \frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j} = 2ky\hat{i}$$

Finding the Stream Function

The two examples above make sense, both mathematically and visually, but there is a subtle difference between them that you probably didn't notice. The first velocity field, $\vec{v} = k\hat{i}$, is "irrotational": that is, $\vec{\nabla} \times \vec{v} = \vec{0}$. The second velocity field does not have a zero curl.

At the beginning of this section we said that we were going to restrict our discussion to irrotational fields. As you will prove in Problem 13.140, the assumption of irrotational fluid flow corresponds to the restriction that $\psi(x, y)$ must be a harmonic function—that is, it must obey Laplace's equation. In the examples above, ky is a solution to Laplace's equation and ky^2 is not.

You may have been wondering what all this is doing in a chapter on complex analysis, and now we are finally ready to make the connection. Under the assumptions we started with—divergenceless, irrotational, two dimensional fluid flow—we can find the stream function by solving Laplace's equation, and then find the velocity field from the stream function. Our strategy for finding harmonic functions is to treat the real plane as if it were the complex plane. (This is why this technique is limited to two dimensions.) We carefully define our boundary conditions. We choose an analytic function, knowing that both its real and imaginary parts must be harmonic. There can be only one harmonic function that fits our boundary conditions, and that is the stream function we're looking for. We can then use Equation 13.10.1 to find the velocity field.

EXAMPLE Return of the Rock in the Stream

We have now built up the tools to analyze the problem that began this section, the stream flowing around a rock in Figure 13.18. We begin as always with our boundary conditions. Since no water flows into or out of the rock, the circular boundary of the rock must be a streamline. By symmetry, the x-axis (the real axis on our complex plane) must also be a streamline. Finally, the water far away from the rock (in any direction) has to look like uniform horizontal flow, which we calculated above as $\psi = ky$.

The streamlines of $\vec{v}$ are the contour lines of ψ, so we need a function $\psi(x, y)$ that is constant on the real axis and on a circle of radius R centered on the origin. For simplicity we can let that constant value be 0. (Remember that we can choose any streamline we want as $\psi = 0$.)

So we are looking for a real function $\psi(x, y)$ that solves Laplace's equation, and that happens to be zero along the border of the rock and the real axis. The obvious choice is $\psi = 0$ (still water) but that does not meet our third boundary condition of horizontal flow far away from the rock, so we have to find something else.

We will begin by choosing an analytic function $f(z)$. Both its real and imaginary parts will be harmonic functions, and we will choose one of them to be our $\psi(x, y)$. On our two contour lines we need $f(z)$ to be a pure imaginary function (and then its real part will be zero) or a pure real function (so its imaginary part is zero).

It's easy to find analytic functions that are real on the real axis, as long as you avoid logs and square roots. (For instance the function $z^2 e^{\sin z}$ is real everywhere on the real axis—obvious when you think about it, isn't it?) The hard part is finding a function that is also real-valued on the edge of the rock.

The boundary of our rock is defined by $|z| = R$, which we can also write as $zz^* = R^2$, so on this circle $z^* = R^2/z$. And now comes a nifty trick: the sum of any function and its complex conjugate is real, so $f(z) = z + R^2/z$ is real on the boundary of the rock. With a little algebra you can find that $Im(f) = y[1 - R^2/(x^2 + y^2)]$.

That's almost the stream function we were looking for. It matches the right boundary condition on the x-axis and the edge of the rock. We know $Im(f)$ is harmonic because $f(z)$ is analytic. (You might notice an exception at $z = 0$ but that point does not concern us at all. Do you see why?) Finally, we need to get $\psi = ky$ far away from the rock. To do that we simply multiply f by the constant k. (Take a moment to convince yourself that this doesn't mess up the other boundary conditions or the fact that f is analytic.)

$$\psi = ky\left(1 - \frac{R^2}{x^2 + y^2}\right) \tag{13.10.3}$$

Finding ψ is the hard part; from there it's easy to find $\vec{v}$ if you want it.

$$\vec{v}(x, y) = \frac{\partial \psi}{\partial y}\hat{i} - \frac{\partial \psi}{\partial x}\hat{j} = k\left(1 - R^2\frac{x^2 - y^2}{(x^2 + y^2)^2}\right)\hat{i} - \frac{2kxyR^2}{(x^2 + y^2)^2}\hat{j}$$

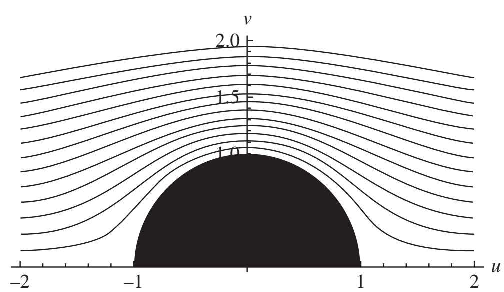
The resulting streamlines are the ones plotted in Figure 13.18. The streamline along the x-axis simply arrives at the obstacle and stops. That streamline is the dividing point between fluid that flows above and below the obstacle, and the point where it reaches the obstacle is called the "stagnation point."

Conformal Mapping and Inverse Problems

Hopefully you were able to follow everything we did in the previous examples, but you might not have come up with the function $f(z) = z + R^2/z$ on your own. Honestly, we probably wouldn't have either. As with electrostatics and steady-state temperature, however, once you know the solutions to a few simple problems you can use conformal mapping to extend those to more complicated ones. And as in those cases, the most useful solutions often come from applying a mapping to a solved problem and then figuring out what harder and hopefully interesting problem you've just solved. In that spirit we end with the following example.

EXAMPLE Flow Around a Complicated Obstacle

Question: The figure below shows a horizontal flow in the upper half-plane going around a semicircular obstacle. From our calculations above we know that the stream function for this flow can be written as $\psi(u, v) = v[1 - 1/(u^2 + v^2)]$. (We are using u and v for our axes because we want to reserve x and y for the more complicated scenario we are going to map this to, and we are setting $k = 1$ for simplicity.) Use the mapping $z(f) = \sqrt{f+1}$ to map this to the stream function for flow around a differently shaped obstacle.



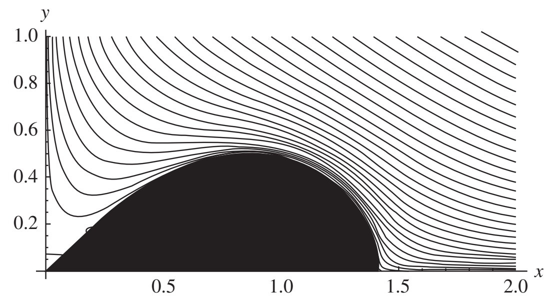
Answer:
There are a number of ways to plot the mapped region. The brute force method is to have a computer take a large array of points in the original region, apply the mapping, and plot their new coordinates. In Problem 13.145 you'll go through analytic calculations to find the mapped region for this problem. Either way it ends up looking like the figure below. The upper half-plane in uv space has mapped to the first quadrant in xy space, and the semicircular obstacle has mapped to a half-teardrop shape.

The process of finding the stream function $\psi(x, y)$ is the same for any conformal mapping problems.

1. *Find the mapping from xy space to uv space.* Since we were given the mapping the other way we need to invert it, giving $f(z) = z^2 - 1$.
2. *Find the real and imaginary parts of the mapping.* This just requires writing $f = u + iv$ and $z = x + iy$, which immediately gives $u(x, y) = x^2 - y^2 - 1$, $v(x, y) = 2xy$.
3. *Plug $u(x, y)$ and $v(x, y)$ into the original stream function.* This gives the final answer.

$$\psi(x, y) = 2xy\left[1 - \frac{1}{(x^2 - y^2 - 1)^2 + 4x^2y^2}\right]$$

The contour lines of this stream function are shown below. This represents flow around a corner with an obstacle.



This example may seem contrived, but the process it illustrates is useful for a variety of problems. In Problem 13.136 you'll use this method to find flow around a corner without a strangely shaped obstacle. In Problem 13.146 you'll use a similar transformation to find the flow around a shape called an "airfoil," which is used to simulate flow around airplane wings.

Stepping Back

The techniques we've used in this section apply to divergenceless, irrotational, laminar fluid flow in two dimensions. ("Laminar" means you can divide the flow into non-interacting streamlines.) Even for that restricted class of flows, there are techniques well beyond what we've covered in this section. In addition to the stream function $\psi(x, y)$ people often calculate a "velocity potential" $\phi(x, y)$ related to $\vec{v}$ by $\vec{\nabla}\phi = \vec{v}$. The functions ψ and ϕ are "harmonic duals," meaning they are the real and imaginary parts of an analytic function $\Omega = \phi + i\psi$, called the "complex potential" of the velocity field. Once you know ψ or ϕ you can find the other one, for example using the Cauchy-Riemann equations, and of course given either one you can find $\vec{v}$. You can use techniques from electrostatics such as the "method of images" to find potential functions for certain flows. Once you have solved a given problem you can reverse the roles of ψ and ϕ (since they are both harmonic) and have the solution to a different fluid flow problem. We don't discuss those methods here, but this section should give you a good introduction to techniques that can be used to solve for fluid flow in certain circumstances.[11]

13.10.1 Problems: Fluid Flow

In all the problems in this section you should assume fluid flow is divergenceless and irrotational unless otherwise specified.

13.134 In the ride "The Cyclone Zone"® at "Wet 'n Wild"® water park in North Carolina riders are carried around by water flowing in circles in a doughnut shaped region between two concentric circles of radius R_1 and R_2. Find a stream function and velocity function for the flow.

[11] For a longer discussion of the use of complex potentials for fluid flow see e.g. "Visual Complex Analysis" by Tristan Needham, Clarendon Press, Oxford, 1997. Chapter 11 introduces the complex potential and Chapter 12, "Fluid Flows and Harmonic Functions"…well, you can figure out what that discusses.

13.135 In the Explanation (Section 13.10.1) we solved for the flow around a circular obstacle, assuming the x-axis was a streamline. If we drop that assumption then any stream function that is constant on the surface of that circle represents a possible flow.

(a) What common analytic function has a constant real part on a circle centered on the origin? Use your answer to find a simple possible stream function around a circular obstacle.

(b) To get a more interesting stream function, add your answer to Part (a) to the stream function we derived in the Explanation. The result will be a new stream function that is harmonic and meets the right boundary conditions. For simplicity you can take $R = k = 1$.

(c) Plot the contours of this new stream function. How is this flow similar to the one we derived in the Explanation and how is it different?

The moral of this story is that the flow around an obstacle can take many forms, depending on the boundary conditions away from the obstacle. That makes sense physically; the same rock in different streams will have different flows around it. To determine the flow for a given physical situation you always need to specify the boundary conditions. When we specified that the flow was horizontal and uniform far from the rock we were led to the unique solution Equation 13.10.3.

13.136 In the boxed example on Page 11 we found the stream function for flow around a corner with an obstacle.

(a) Repeat the process to find the stream function for flow around a corner without the obstacle present.

(b) Sketch the contour lines of the stream function (first quadrant only).

13.137 In the Explanation (Section 13.10.1) we solved for flow bounded by the horizontal axis and a circle centered on the origin.

(a) Use the mapping $z(f) = f^{1/3}$ to map this stream function to a more complicated flow problem.

(b) Find the velocity field $\vec{v}(x, y)$.

(c) Sketch the region and the contour lines of $\psi(x, y)$.

(d) Sketch the region and the vector field $\vec{v}(x, y)$.

13.138 Horizontal flow in the upper half of the uv-plane is described by the stream function $\psi = v$, which is the imaginary part of the analytic function $f = u + iv$. In this problem you're going to use the mapping $z(f) = \sqrt{f^2 - 1}$ to map this simple solution to a more interesting one.

(a) What region in the xy-plane is the boundary $v = 0$ mapped to? In words, what fluid flow problem are you solving with this mapping?

(b) Find the stream function $\psi(x, y)$. The following identity may help.

$$\sqrt{a + bi} = \frac{1}{\sqrt{2}}\sqrt{a + \sqrt{a^2 + b^2}} + \frac{i}{\sqrt{2}}\sqrt{-a + \sqrt{a^2 + b^2}}$$

(c) Plot contours of this new stream function. This solution describes horizontal flow around a boundary. Describe the shape of that boundary.

13.139 In the Explanation (Section 13.10.1) we solved for flow around a circular barrier. Use the mapping $z(f) = (1/2)(3f + 1/f)$ to map this problem to a more complicated one. Sketch the mapped region, find the mapped stream function, and sketch its contours. *Nothing in this problem requires a computer, but the algebra and the sketching are kind of ugly without one.*

13.140 Assume a velocity field is irrotational ($\vec{\nabla} \times \vec{v} = \vec{0}$) and prove that the stream function $\psi(x, y)$ is harmonic.

13.141 We defined the stream function $\psi(z)$ as the flux through a curve drawn from a reference point z_0 to the point z.

(a) Sketch a region with two points z_0 and z and draw two different curves connecting them. Use the divergence theorem to argue that the flux through the two curves must be the same if $\vec{\nabla} \cdot \vec{v} = 0$ everywhere in the region, and will not generally be the same otherwise.

(b) Suppose you have defined a stream function $\psi(z)$ using a reference point z_0, but then you change your mind and decide to define a new stream function $\chi(z)$

using a different reference point z_1. Write an equation relating $\psi(z)$ and $\chi(z)$.

13.142 The Explanation (Section 13.10.1) described the relationship between the stream function ψ and the velocity field $\vec{v}$ in several ways that are not obviously connected to each other. In this problem you will draw the key connection by starting with Equation 13.10.2 and proving Equation 13.10.1 (up to the sign ambiguity mentioned in the Explanation).

(a) Explain why $\vec{\nabla}\psi$ must be perpendicular to $\vec{v}$.

(b) Next you need to relate the magnitude of $\vec{\nabla}\psi$ to the magnitude of $\vec{v}$. Consider two streamlines separated by a line segment of length ds. If the streamlines are close enough, that line segment can be perpendicular to both streamlines. Let $d\psi$ be the difference in the values of ψ between the two streamlines.

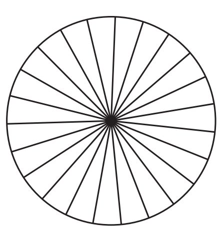
i. Express $d\psi$ in terms of ds and the magnitude of $\vec{\nabla}\psi$.

ii. Express $d\psi$ in terms of ds and the magnitude of $\vec{v}$.

iii. How are the magnitudes of $\vec{\nabla}\psi$ and $\vec{v}$ related?

(c) Write a vector that is perpendicular to $\vec{\nabla}\psi$ and has the same magnitude as $\vec{v}$. Your answer should only contain ψ, not $\vec{v}$. *Hint*: recall that two vectors $c\hat{i} + d\hat{j}$ and $d\hat{i} - c\hat{j}$ are perpendicular to each other.

13.143 Water is falling onto the middle of a circular table and flowing out to the edge at radius R. One boundary condition is that the streamlines must be normal to the edge of the table. The other is that the streamlines must point directly outward from the point at the center. (If that's a "boundary condition" then what is the "boundary"? Imagine the water is coming straight out of a small circular area of radius r in the center, and then take the limit of your final answer as $r \to 0$.)



(a) Find a harmonic function that meets these boundary conditions and thus write the stream function for this flow. Your answer should have one undetermined constant in it.

(b) Find $\vec{v}$.

(c) Water has a density of 1 g/cm^3. If the layer of water on the table is 1 cm thick then its mass per unit *area* on the table is 1 g/cm^2. Assuming water is being added to the center at 5 kg/s, find the constant in your expression for $\vec{v}$.

13.144 *[This problem depends on Problem 13.143.]* In Problem 13.143 you found the fluid flow on a circular table with a source at the center.

(a) Verify that this flow is divergenceless everywhere except at the origin.

(b) Argue using the divergence theorem that the divergence cannot be zero at the origin.

Because the domain in which the flow is divergenceless isn't "simply connected" (meaning there's a hole in the middle of it), the stream function is not single-valued. To demonstrate this, consider the points $P = (1/2, 0)$ and $Q = (0, 1/2)$.

(c) Draw an arc going counterclockwise from P to Q. Find the flux through this arc.

(d) Draw an arc going clockwise from P to Q. Find the flux through this arc.

(e) Take P as the reference point for the stream function. Find ψ if the curve you draw from P to any point z is always a combination of a counterclockwise arc and a radial line segment.

(f) Using the same reference point, find ψ assuming each curve is a combination of a clockwise arc and a radial line segment.

(g) Show that these two stream functions describe the same flow. (Your two velocity functions may differ in sign because of the direction ambiguity of flux through a non-closed curve, but they should otherwise be identical.)

13.145 In the boxed example on Page 11 we used the mapping $z(f) = \sqrt{f+1}$ to map flow in the upper half-plane around a semicircular obstacle to a more complicated problem. In this problem you'll work out the mapping we used there.

(a) What region does the transformation $z = f + 1$ map the upper half-plane to? *Hint*: This is trivial if you think about it.

(b) Now apply the transformation $z = \sqrt{f}$ to the region you found in Part (a) to find the region that $z = \sqrt{f+1}$ maps the upper half-plane to. In answering this part you should assume you are using the principal branch of the square root function.

(c) The lower boundary of the semicircle is the line segment from $(-1, 0)$ to $(1, 0)$. What does this line segment get mapped to?

(d) The top boundary of the semicircle satisfies $u^2 + v^2 = 1$. Using the formulas for $u(x, y)$ and $v(x, y)$ we derived in the example, write an equation for the curve in xy space that defines the upper boundary of the new obstacle.

(e) Make a sketch of the curve you just found. You can do this with a computer or by hand, but if you do it by hand explain how you know the basic shape.

13.146 **Exploration: The Joukowski Airfoil**
In the Explanation (Section 13.10.1) we argued that the function $z + R^2/z$ is real on a circle of radius R centered on the origin. An equivalent way of saying that is that the mapping $z(f) = f + R^2/f$ maps this circle to part of the x-axis. In 1908 Nikolai Joukowski applied this mapping to circles with other centers and found that they produced interesting shapes. You'll work with one such example here.

(a) Apply the mapping $z = f + 1/f$ to a circle that passes through the point $f = -1$ but is centered on $f = .1 + .2i$. Plot the resulting shape. This shape is called an "airfoil," and is often used to model airplane wings.[12]

(b) Invert the mapping to find $f(z)$. You should get two solutions. For now you'll just hold onto both of them.

For each of the two inverse functions $f(z)$ you found, have the computer define functions $u(x, y)$ and $v(x, y)$ and use them to define a function $\psi(x, y)$. *Don't copy this function down. You don't even have to print it on your screen. It's pretty ugly.*

(c) Make two plots showing the contours of $\psi(x, y)$ around the airfoil, one for each stream function you defined. You should find that each of them fits around the airfoil perfectly in some parts of the plot and not in others.

(d) Define a new function equal to the appropriate stream function in each part of the domain. Plot the contours of this piecewise function around the airfoil to see the flow. You should be able to see the stagnation point.

[12] See e.g. *Theoretical Aerodynamics, 4th ed.* by L. M. Milne-Thomson, Macmillan and Company, 1966.