12.11 Additional Problems

For Problems 12.154–12.159 solve the differential equation using either the method of power series or the method of Frobenius. Your answer should be in the form of a partial sum with five non-zero terms and two arbitrary constants.

12.154 $4y''(t) + y'(t) + t^3 y(t) = 0$

12.155 $y''(t) + (1+t)y(t) = t$

12.156 $ty''(t) + 2y'(t) + y(t) = 0$. (Your answer will only have one arbitrary constant, and will therefore not be the general solution.)

12.157 $y''(t) + y'(t) + y(t) = \sin t$

12.158 $e^t y''(t) + y(t) = 0$

12.159 $y''(t) + (3/2)(\cos t/t)y'(t) + y(t) = 0$ (You may just report the first four non-zero terms for this one.)

12.160 **Airy Functions.** The "Airy equation" arises in the quantum mechanical treatments of a triangular potential well and for a particle in a one-dimensional constant force field.

$$\frac{d^2y}{dx^2} - xy = 0$$

(a) Explain how we can tell by looking at this equation that a power series solution exists.

(b) Find the recurrence relation using the method of power series.

(c) What coefficients must be zero?

(d) Write the solution up to the seventh order for $c_0 = c_1 = 1$.

12.161 **Laguerre Polynomials.** The "Laguerre differential equation" appears in the quantum mechanical treatment of the Hydrogen atom.

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$$

(a) Show that the Laguerre differential equation meets the requirements for the method of Frobenius, but does not meet the requirements for the method of power series.

(b) Begin applying the method of Frobenius. Stop after you reach and solve the indicial equation.

Tragedy strikes! You discover that r must be zero which means the solution will in fact be a power series. (The moral of that story is that analytic coefficients are sufficient, but not necessary, for the method of power series.) So we begin again.

(c) Assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$, find the recurrence relation for c_{n+1} in terms of c_n, n, and the constant λ.

(d) For any positive integer λ the resulting solution will be a finite polynomial. How can we tell this, and what will be the order of the polynomial?

(e) Find a solution for $\lambda = 0$ and $c_0 = 1$. This is the first "Laguerre polynomial" generally denoted $L_0(x)$.

(f) Find a solution for $\lambda = 1$ and $c_0 = 1$. This is the next "Laguerre polynomial" $L_1(x)$.

(g) Find $L_2(x)$ and $L_3(x)$.

(h) Look up the Laguerre polynomials. If the first four do not agree with your answers, figure out what went wrong!

The power series method only gave us one solution. More complicated methods can be used to find the other one, but it isn't typically relevant for physical applications.

12.162 **Exploration: The Quantum Harmonic Oscillator**

A harmonic oscillator is defined by the potential function[15] $V = (1/2)m\omega^2 x^2$. The quantum mechanical wavefunction for such an oscillator must obey Schrödinger's equation with that potential function.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

If you've never heard of a wavefunction, don't let it scare you off: this is just a differential equation for a function that happens to be called $\psi(x)$. The constants m and ω correspond to a particular physical situation, and $\hbar$ is a universal constant. Your job is to find the values of the constant E for which solutions exist (the eigenvalues of this equation), and the solutions associated with them (the eigenfunctions). (Section 12.10

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[15]You may be more used to the form $V = (1/2)kx^2$. Since ω is defined to be $\sqrt{k/m}$ the two are equivalent, but this form is more commonly used in quantum problems.

shows you how to solve this problem in a very different way from what we do here.)

(a) You can rewrite this ODE in the form $\psi''(y) - (y^2 - \lambda)\psi(y) = 0$ with a substitution of the form $y = cx$. Find the constants c and λ in terms of $\hbar$, m, ω, and E. What are the units of y and λ?

(b) We are going to guess that ψ can be written as $e^{-y^2/2}$ times a simpler function. (That guess can be physically motivated, but we're going to skip that step here. The real justification will be to try it and see if the resulting solution looks simple.) Using the substitution $\psi(y) = e^{-y^2/2}u(y)$, rewrite the differential equation.

(c) Solve this equation using the method of power series. Your answer should be in the form of a recurrence relation for the coefficients.

The recurrence relation gives two series, one with the even coefficients c_0, c_2, etc., and another with the odd coefficients. For any given value of λ, the coefficients c_0 and c_1 are the two arbitrary constants in the general solution, and in general each one is the beginning of an infinite series. However, non-terminating polynomials do not represent physically meaningful solutions to this problem.[16]

(d) For what values of λ will one of the two series be a *finite* polynomial, meaning all the coefficients beyond some value of n will equal zero? (Your answer gives the eigenvalues of this problem.)

(e) For the three lowest eigenvalues λ, write the solutions $\psi(x)$. Each one will be in the form of a finite polynomial (with an arbitrary constant) times an exponential.

(f) The constant E represents the oscillator's energy. Based on the allowed values of λ that you found, what are the possible values of E?

12.163 Exploration: A Different Kind of Series Solution

The following differential equation describes the motion of an object falling in the gravitational field of a planet.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{12.11.1}$$

Here r is the distance of the object from the center of the planet. The given constants in this problem are G (a universal constant), M (the mass of the planet, not of the falling object), and r_0 and v_0 (the initial position and velocity of the object).

(a) Explain why this equation does not lend itself directly to either the power series method, or the method of Frobenius, as discussed in this chapter.

Nonetheless, we can approach this problem by looking for the first few coefficients in a Maclaurin series solution:

$$r(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \dots \tag{12.11.2}$$

(b) Using Equation 12.11.2 find the first two coefficients in terms of the given constants.

(c) Take the second derivative of both sides of Equation 12.11.2. Then use Equation 12.11.1 to replace d^2r/dt^2, and finally plug 0 into both sides to find c_2 in terms of the given constants.

(d) Write the solution to Equation 12.11.1 up to the second order. This solution should look like the introductory mechanics equation $x = x_0 + v_0 t - (1/2)gt^2$ with the constant g being a function of our given constants. (This g should come out as 9.8 m/s^2 if you use the mass and radius of the Earth as M and r_0.)

(e) To find the next term—the first correction to the introductory mechanics equation—take the derivative of both sides of Equation 12.11.1 with respect to time, and then plug in $t = 0$. Solve for c_3 in terms of the given constants.

(f) Write the solution to Equation 12.11.1 up to the third order.

(g) We have seen that your second-order formula replicates the introductory mechanics equation $r = r_0 + v_0 t - (1/2)gt^2$, which works well for objects that stay near the surface of the Earth ($r \approx R$). Does the third-order correction make the effective acceleration due to gravity higher, or lower, than 9.8 m/s^2? Answer based on your equation, but then explain why your answer makes sense physically. Assume

[16] We're not going to go through the proof of that claim here. See for example "Introduction to Quantum Mechanics" by David Griffiths (one of our personal heroes).

$r_0 = R$ and $v_0 > 0$ (such as a rocket taking off). (Your answer will depend on time.)

(h) A bullet is fired straight up into the air from the surface of the Earth with an initial speed of 1000 m/s. Look up values for G, M, and r_0, and use them to graph the height of the bullet using your answers to Parts (d) and (f). How does adding the third-order term change the motion of the bullet?