12.10 Special Application: The Quantum Harmonic Oscillator and Ladder Operators

In this section we're going to derive the possible states of a particle in a potential field $V = (1/2)kx^2$, an important problem in quantum mechanics. To do that we have to introduce the technique of "ladder operators." To do *that* we have to spend some time on the notation of operators, and especially on the idea of a "commutator." As usual, the math you pick up along the way will apply to a variety of different physical situations.

If you haven't worked much with differential operators we recommend starting with the Discovery Exercise, which will give you some practice with them.

What we are not going to do in this section is explain the basics of quantum mechanics: Schrödinger's equation, eigenstates and energy levels, normalization, and so on. If you have never seen any quantum mechanics this section may assume too much for you. If you are particularly interested we have two papers on the subject. (We highly recommend reading the first one before the second one.)

- `http://www.felderbooks.com/papers/quantum.html` is a non-mathematical introduction to some fundamental ideas of quantum mechanics, how those ideas radically depart from classical physics, and why this radical departure is necessitated by experimental results.
- `http://www.felderbooks.com/papers/psi.html` is an overview of the mathematical approach required to solve quantum mechanical problems. If you got lost somewhere between the second-order PDE, the Fourier transform, and the eigenstates, this may help you see how they fit together.

12.10.1 Discovery Exercise: The Quantum Harmonic Oscillator and Ladder Operators

The derivative d/dx is called an "operator," meaning it takes as input a function and produces as output another function. We will abbreviate that derivative as D. The operator x just multiplies any function by x. So for example, the operator $D - x$ acting on the function $f(x) = x^2$ produces the function $(D - x)f = 2x - x^3$. For this exercise we define the operators $\hat{a}_R = D - x$ and $\hat{a}_L = D + x$.

1. Calculate $\hat{a}_L \sin x$. (This should be very simple; we just want to make sure you're clear on the notation.)

   See Check Yourself #88 in Appendix L

2. Calculate $\hat{a}_L \hat{a}_R x^2$. (Read this as "act with $\hat{a}_R$ on x^2, then act with $\hat{a}_L$ on the result.")

The easiest way to do algebra with operators is to see what they do to an arbitrary function. For example, the operator Dx acting on a function f gives $(d/dx)(xf) = x(df/dx) + f$. We can then take out the f and write $Dx = xD + 1$, where the 1 represents the operator "multiply the function by 1."

3. Calculate $\hat{a}_L \hat{a}_R$. *Hint*: a second derivative is D^2 in operator notation.

   See Check Yourself #89 in Appendix L

4. The "commutator" of two operators $\hat{A}$ and $\hat{B}$ is defined as $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Calculate the commutator $\left[\hat{a}_L, \hat{a}_R\right]$.

12.10.2 Explanation: The Quantum Harmonic Oscillator and Ladder Operators

A particle that experiences a force $F = -kx$ is a simple harmonic oscillator. Its potential energy is $V = (1/2)m\omega^2 x^2$ where m is the particle's mass and $\omega = \sqrt{k/m}$. The quantum mechanical wavefunction for such an oscillator must obey Schrödinger's equation with that potential function.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \tag{12.10.1}$$

Here $\hbar$ is a constant of nature and m and ω are both constant for a particular physical system. The energy E, however, can take on different values. The solution $\psi(x)$ for any particular E is the wavefunction for the state where the particle has that value of energy. Classically the particle could have any energy from 0 to ∞, but quantum mechanically only certain values of E are possible. Those are the eigenvalues of Equation 12.10.1 subject to the boundary conditions that $\psi(x)$ must be finite at $x \to \pm\infty$.

In this section we will find the eigenvalues E and corresponding eigenfunctions $\psi(x)$ that represent the energy states of the quantum harmonic oscillator. In Section 12.11 Problem 12.162 you'll solve this problem using the method of power series, but it involves a few subtleties. In this section we'll solve it in a very different way, using the so-called "ladder operators" developed by Paul Dirac.

You'll show in Problem 12.148 that the substitutions $y = \sqrt{m\omega/\hbar}\, x$ and $\lambda = 2E/(\hbar\omega)$ let us rewrite Equation 12.10.1 as:

$$\frac{d^2\psi}{dy^2} - y^2\psi + \lambda\psi = 0 \tag{12.10.2}$$

We're going to solve this problem in two steps. First we'll show you a trick that will allow us to find the eigenfunctions and eigenvalues quickly and easily. In that part we'll present this trick with no justification, as if it had been handed to Dirac on stone tablets. Later we'll show where the trick came from, which will suggest how you might apply it to other problems.

The Trick with No Justification

Suppose $\psi_m(y)$ is an eigenfunction of Equation 12.10.2 with eigenvalue $\lambda = c_m$. One day, for no obvious reason, Paul Dirac asks us to see if $\psi_n = d\psi_m/dy + y\psi_m$ happens to be an eigenfunction of the same equation. Agreeably, we begin by finding its first derivative.

$$\psi_n = \frac{d\psi_m}{dy} + y\psi_m \quad \rightarrow \quad \frac{d\psi_n}{dy} = \frac{d^2\psi_m}{dy^2} + y\frac{d\psi_m}{dy} + \psi_m$$

Now remember that ψ_m is an eigenfunction of the differential equation with eigenvalue c_m, so we know that $d^2\psi_m/dy^2 = y^2\psi_m - c_m\psi_m$.

$$\begin{aligned}
\frac{d\psi_n}{dy} &= y^2\psi_m - c_m\psi_m + y\frac{d\psi_m}{dy} + \psi_m \\
&= y\frac{d\psi_m}{dy} + \left(y^2 - c_m + 1\right)\psi_m \\
\frac{d^2\psi_n}{dy^2} &= y\frac{d^2\psi_m}{dy^2} + \left(y^2 - c_m + 2\right)\frac{d\psi_m}{dy} + 2y\psi_m \\
&= \left(y^2 - c_m + 2\right)\frac{d\psi_m}{dy} + \left(y^3 + 2y - yc_m\right)\psi_m
\end{aligned}$$

Now plug that into $d^2\psi_n/dy^2 - y^2\psi_n$. If ψ_n is an eigenfunction then the result should be a constant (minus the eigenvalue) multiplied by ψ_n.

$$\frac{d^2\psi_n}{dy^2} - y^2\psi_n = \left(-c_m + 2\right)\frac{d\psi_m}{dy} + \left(2y - yc_m\right)\psi_m = -(c_m - 2)\left(\frac{d\psi_m}{dy} + y\psi_m\right) = -(c_m - 2)\psi_n$$

We've therefore shown that the function ψ_n is in fact an eigenfunction of Equation 12.10.2, and its eigenvalue is $c_m - 2$. In Problem 12.149 you'll go through a similar calculation to show that $d\psi_m/dy - y\psi_m$ is also an eigenfunction, with eigenvalue $\lambda = c_m + 2$. So all we need to do is find one eigenfunction and these two operators will generate as many more as we wish.

But Sturm-Liouville theory says there should be a lowest eigenvalue. If we keep applying $d\psi_m/dy + y\psi_m$ it seems that we will keep finding different eigenfunctions, each with a lower eigenvalue than the one before. The series will only terminate if we find that $d\psi_m/dy + y\psi_m$ gives us zero; then there will not be another valid eigenfunction, so there will be no lower eigenvalues.

Conveniently that fact lets us find the lowest eigenfunction. The solution to $d\psi/dy + y\psi = 0$ is $\psi = C_0 e^{-y^2/2}$, so that is the state with the lowest eigenvalue. In Problem 12.150 you'll derive that solution and show that its eigenvalue is $\lambda = 1$.

If we apply our trick to find the next eigenfunction we get $\psi = (d/dy)(C_0 e^{-y^2/2}) - yC_0 e^{-y^2/2} = -2C_0 y e^{-y^2/2}$ with eigenvalue $\lambda = 3$. Each eigenfunction can have a different arbitrary constant in front of it, however, so we replace $-2C_0$ with a new constant C_1.

The remaining solutions will have eigenvalues 5, 7, 9, and so on. The last step is to convert back to our original variables x and E instead of y and λ. Any constants that appear in front of the functions can be absorbed into the arbitrary constants, and the resulting first few eigenfunctions are as follows.

$\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$	$E = (1/2)\hbar\omega$	The ground state
$\psi_1(x) = C_1 x e^{-m\omega x^2/2\hbar}$	$E = (3/2)\hbar\omega$	The first excited state
$\psi_2(x) = C_2[(2m\omega/\hbar)x^2 - 1]e^{-m\omega x^2/2\hbar}$	$E = (5/2)\hbar\omega$	The second excited state

The constants C_n are determined by the "normalization condition" $\int_{-\infty}^{\infty}\psi(x)^2\,dx = 1$. See Problem 12.151.

Reframing the Problem in the Language of Operators

To see where that trick came from we need to talk about the problem in terms of "operators." We introduced operators in Chapter 10 but we'll recap the key ideas here. Becoming comfortable with operators will be more useful in the long run than anything you learn about quantum oscillators.

An operator acts on a function to produce another function. For instance if D is the operator "take the derivative with respect to y" then we can write $D(y^3) = 3y^2$.

When a constant is used as an operator it indicates multiplication. So the operator 7 turns y^3 into $7y^3$, and the operator 1 turns any function into itself.

All the operators that concern us in this section will be "linear operators." An operator $\hat{A}$ is linear if it obeys the following two rules.

- $\hat{A}(f + g) = \hat{A}f + \hat{A}g$
- $\hat{A}(kf) = k\hat{A}f$

The product of two operators $\hat{A}\hat{B}$ is defined as the operator "do $\hat{B}$, then $\hat{A}$." It's important to go right to left, just as with matrices, and for the same reason: so that $(\hat{A}\hat{B})f$ is the same function as $\hat{A}(\hat{B}f)$. (Formally this means that operator multiplication is "associative." Informally it means we can put parentheses anywhere we like in a string of operator multiplications.)

EXAMPLE **Operator Multiplication**

Define the operator D to mean "take the derivative with respect to y." The operator y means "multiply the function by y."

Question: What do the operators yD and Dy do to a function $f(y)$?

Answer:

$$yDf = y\left(\frac{df}{dy}\right)$$
$$Dyf = \frac{d}{dy}(yf) = y\left(\frac{df}{dy}\right) + f$$

We see that $Dyf = yDf + f$, an equation that relates two functions. We can rewrite that as an equation that directly relates two operators: $Dy = yD + 1$. (Remember what the operator "1" means!)

The example above illustrates the very general fact that operator multiplication is not commutative. If you want to switch the order of an operator multiplication you need to use a "commutator," as we discuss below. Note also that, because of this definition of operator multiplication, squaring an operator means doing that operator twice. So D^2 gives a second derivative, not a first derivative squared.

With this terminology we can rewrite Equation 12.10.2 as $(D^2 - y^2)\psi + \lambda\psi = 0$. We left the λ term separate to make it clear that this equation is asking for the eigenvalues and eigenfunctions of the operator $D^2 - y^2$. If these were numbers we could factor that into $(D + y)(D - y)$, or equivalently $(D - y)(D + y)$. Since these are operators instead of numbers those two expressions are not equivalent, and in fact neither one gives $D^2 - y^2$.

Let's see what they do give us. Since operators are defined by how they act on functions, it's easiest to manipulate them by putting a function after them, so we'll consider how these operators act on an arbitrary function $f(y)$.

$$\begin{aligned}(D+y)(D-y)f &= \left(\frac{d}{dy} + y\right)\left(\frac{df}{dy} - yf\right) = \frac{d^2f}{dy^2} - \frac{d}{dy}(yf) + y\frac{df}{dy} - y^2f \\ &= \frac{d^2f}{dy^2} - y\frac{df}{dy} - f + y\frac{df}{dy} - y^2f = (D^2 - 1 - y^2)f\end{aligned}$$

We can now drop the f and write an operator equation.

$$(D + y)(D - y) = (D^2 - y^2 - 1)$$

In Problem 12.141 you'll do a similar calculation to show that $(D - y)(D + y) = (D^2 - y^2 + 1)$.

The key ideas in this section flow from calculations like the one above, so we urge you to go through it carefully. The -1 in our final result came from the difference between "multiplying-by-y-and-then-taking-a-derivative" and "taking-a-derivative-and-then-multiplying-by-y." The order of operator multiplication usually matters, as it does in this case. When you want to reverse the order you need the "commutator."

Definition: Commutator

The "commutator" of two operators is the difference between multiplying them in different orders.

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}$$

If two operators "commute" (that is, $\hat{A}\hat{B}=\hat{B}\hat{A}$) then their commutator is zero. You should be able to easily convince yourself that every operator commutes with itself: $[\hat{A},\hat{A}]=0$. The second rule we gave above for linear operators can be expressed as "a linear operator commutes with a constant operator."

The list below summarizes some of the important arithmetic properties of linear operators.

Property	Name	Why it Works
$(\hat{A}\hat{B})\hat{C}=\hat{A}(\hat{B}\hat{C})$	Associative Property	This is true for all operators, because $\hat{A}\hat{B}$ is defined as the operator that makes this true.
$(\hat{A}+\hat{B})\hat{C}=\hat{A}\hat{C}+\hat{B}\hat{C}$	Distributive Property	This is true for all operators, because $\hat{A}+\hat{B}$ is defined as the operator that makes this true.
$\hat{A}(\hat{B}+\hat{C})=\hat{A}\hat{B}+\hat{A}\hat{C}$	Distributive Property	This is part of the definition of a linear operator.
$\hat{A}k=k\hat{A}$	Commuting with a constant	This is part of the definition of a linear operator.
$\hat{A}\hat{B}=\hat{B}\hat{A}+\left[\hat{A},\hat{B}\right]$	Commutator	This is the definition of the commutator.

The following example shows how to use these properties. Operator arithmetic is a useful skill in general, but the particular operators in the example will also be important for our purposes.

EXAMPLE **Commutator**

Let $\hat{a}_L=D+y$ and $\hat{a}_R=D-y$. (The reasons for the subscripts R and L will become clear later.)

Problem:
Find the commutator $\left[\hat{a}_R,\hat{a}_L\right]$.

Solution:
We found $\hat{a}_L\hat{a}_R$ above, and you will find $\hat{a}_R\hat{a}_L$ in Problem 12.141. All that remains is to subtract them.

$$\left[\hat{a}_R,\hat{a}_L\right]=\hat{a}_R\hat{a}_L-\hat{a}_L\hat{a}_R=2$$

That means that we can replace $\hat{a}_R\hat{a}_L$ in any equation with $\hat{a}_L\hat{a}_R+2$. (Every time an $\hat{a}_R$ passes right through an $\hat{a}_L$ it picks up a +2.) Equivalently, we can replace $\hat{a}_L\hat{a}_R$ with $\hat{a}_R\hat{a}_L-2$. (Every time an $\hat{a}_R$ passes left through an $\hat{a}_L$ it picks up a −2.)

Problem:
Rewrite $\hat{P}=\hat{a}_L\hat{a}_L\hat{a}_R$ with the $\hat{a}_R$ on the left.

Solution:
Pay careful attention to what operator properties we use at each step of this solution.

$$\begin{aligned}\hat{P} &= \hat{a}_L(\hat{a}_L\hat{a}_R)\\ &= \hat{a}_L(\hat{a}_R\hat{a}_L - 2)\\ &= \hat{a}_L\hat{a}_R\hat{a}_L - \hat{a}_L 2\\ &= (\hat{a}_L\hat{a}_R)\hat{a}_L - 2\hat{a}_L\\ &= (\hat{a}_R\hat{a}_L - 2)\hat{a}_L - 2\hat{a}_L\\ &= \hat{a}_R\hat{a}_L\hat{a}_L - 2\hat{a}_L - \hat{a}_L 2\\ &= \hat{a}_R\hat{a}_L\hat{a}_L - 4\hat{a}_L\end{aligned}$$

If it's useful for your calculations you can rewrite this as $\hat{P} = (\hat{a}_R\hat{a}_L - 4)\hat{a}_L$.

Raising and Lowering Operators

We now return to Equation 12.10.2. Since we found that $(D + y)(D - y) = (D^2 - y^2 - 1)$ we can rewrite this equation in terms of our operators $\hat{a}_L$ and $\hat{a}_R$.

$$(\hat{a}_L\hat{a}_R + 1)\psi + \lambda\psi = 0 \tag{12.10.3}$$

We found above that $\left[\hat{a}_R, \hat{a}_L\right] = 2$ so we will replace $\hat{a}_R\hat{a}_L$ with $\hat{a}_L\hat{a}_R + 2$ in the middle of the following calculation, leading us to the same equation with a *different* eigenvalue.

$$\begin{aligned}(\hat{a}_L\hat{a}_R + 1)(\hat{a}_L\psi_m) + \lambda(\hat{a}_L\psi_m) &= \hat{a}_L\hat{a}_R\hat{a}_L\psi_m + (1+\lambda)\hat{a}_L\psi_m\\ &= \hat{a}_L(\hat{a}_L\hat{a}_R + 2)\psi_m + (1+\lambda)\hat{a}_L\psi_m\\ &= \hat{a}_L(\hat{a}_L\hat{a}_R\psi_m) + (3+\lambda)\hat{a}_L\psi_m\end{aligned}$$

We now use the fact that ψ_m is an eigenfunction to replace $\hat{a}_L\hat{a}_R\psi_m$ with $(-1 - c_m)\psi_m$, which gives:

$$(2 - c_m + \lambda)\hat{a}_L\psi_m = (2 - c_m + \lambda)\psi_n$$

We conclude that ψ_n is an eigenfunction with eigenvalue $\lambda = c_m - 2$, and a similar calculation leads us to conclude that $\hat{a}_R\psi_m$ is an eigenfunction with eigenvalue $\lambda = c_m + 2$. (See Problem 12.149.)

The operators $\hat{a}_R$ and $\hat{a}_L$ are called the "raising" and "lowering" operators for this problem. (Hence the subscripts.) The raising operator takes any eigenfunction and turns it into the one with the next highest eigenvalue, and the lowering operator turns it into the one with the next lowest eigenvalue. When you act with the lowering operator on the ground state you get 0, which leads to the alternative names "annihilation" and "creation" operators.[13] Together, $\hat{a}_R$ and $\hat{a}_L$ are called "ladder operators" because they generate a ladder of states with eigenvalues stretching upward from the ground state.

[13]These names make more sense in quantum field theory, where the eigenvalue represents the number of particles. Then the annihilation operator takes you from a state with n particles to one with $n - 1$, and the creation operator takes you to one with $n + 1$ particles.

Stepping Back

We just solved the quantum oscillator problem twice, once by manipulating differential equations and then again in the more abstract language of operators.[14] You may reasonably feel that the second approach made the problem look more confusing without giving you anything new, so we want to point out a few reasons why the operator approach is in the long run better. It's worth putting in the effort to learn to work with operators now, since they are used widely in quantum mechanics, optics, and a variety of other fields.

For one thing, operators make the calculations easier. Of course the operators are just shorthand for pieces of differential equations, but once you rewrite the equation in terms of $\hat{a}_R$ and $\hat{a}_L$ and calculate their commutator you've reduced a calculus problem to a simple algebra problem.

More importantly the operator formulation shows *why* this strange trick worked, and that allows you to apply it to other problems. When we plugged $\hat{a}_R\psi_m$ into Equation 12.10.3 we had to move the $\hat{a}_R$ to the left of $\hat{a}_L\hat{a}_R$ so we could simplify by acting with $\hat{a}_L\hat{a}_R$ on the eigenfunction ψ_m. Because $[\hat{a}_L, \hat{a}_R]$ is a constant, the result was to simply add a constant to the eigenvalue in the equation. In Problem 12.152 you'll try this trick on a similar problem where the commutator of the two operators is not a constant, and you'll see that it doesn't work. Problem 12.153 is another problem where it *does* work, finding the angular momentum values of the hydrogen atom.

Finally, a warning: if you take a course on quantum mechanics, expect to encounter different conventions from those we used here. The conventional ladder operators, usually denoted $\hat{a}_+$ and $\hat{a}_-$, are related to ours by $\hat{a}_- = \hat{a}_L/\sqrt{2}$ and $\hat{a}_+ = -\hat{a}_R/\sqrt{2}$. We therefore ended up with the commutator $[\hat{a}_R, \hat{a}_L] = 2$ rather than the more common $[\hat{a}_-, \hat{a}_+] = 1$.

12.10.3 Problems: The Quantum Harmonic Oscillator and Ladder Operators

The problems in this section assume the following definitions.

$$D = d/dy \qquad \hat{a}_R = D - y \qquad \hat{a}_L = D + y$$

You will also make frequent use of the following fact (derived in the Explanation above).

$$\left[\hat{a}_R, \hat{a}_L\right] = 2 \quad \rightarrow \quad \hat{a}_R\hat{a}_L = \hat{a}_L\hat{a}_R + 2$$

This means that you can pass $\hat{a}_R$ right through $\hat{a}_L$ and pick up a +2, or pass $\hat{a}_R$ left through $\hat{a}_L$ and pick up a −2.

[14]Remember when the differential equations themselves seemed hopelessly abstract? Ah, those innocent bygone days.

12.139 In the box on Page 8 we step through the process of rewriting the operator $\hat{a}_L\hat{a}_L\hat{a}_R$ with $\hat{a}_R$ on the left. Copy that solution (only the part about the three operators) and say what operator property justifies each step. *Hint*: None of the steps depends in any way on knowing what the operators $\hat{a}_L$ and $\hat{a}_R$ are. Some steps depend on knowing that both are linear operators, and other steps depend on knowing their commutator.

12.140 **Operator Algebra** Commutators are used to pass one operator "through" another, as in the example that starts on Page 8.

(a) Rewrite each of the following expressions so every $\hat{a}_R$ is to the right of every $\hat{a}_L$.

i. $\hat{a}_R\hat{a}_L\hat{a}_R$

ii. $\hat{a}_R\hat{a}_L\hat{a}_L$

(b) Suppose ψ is an eigenfunction of the operator $\hat{a}_R$ with eigenvalue λ, which means $\hat{a}_R\psi = \lambda\psi$. Use that fact and the commutator above to simplify the expression $\hat{a}_R\hat{a}_L\psi$ as much as possible.

(c) Suppose ϕ is an eigenfunction of the operator $\hat{a}_R\hat{a}_L$ with eigenvalue γ. Is ϕ an eigenfunction of $\hat{a}_L\hat{a}_R$? If not, explain why not. If so, find its eigenvalue.

12.141 Calculate each of the following. Give your answers in terms of D and y, not $\hat{a}_R$ and $\hat{a}_L$.

(a) $[D, y]$

(b) $\hat{a}_R^2$

(c) $\hat{a}_R\hat{a}_L$

12.142 Calculate each of the following. Simplify your answers as much as possible. You should be able to answer all of these just from knowing the commutator $[\hat{a}_R, \hat{a}_L]$, without having to use the definitions of $\hat{a}_R$ and $\hat{a}_L$.

(a) $[\hat{a}_L, \hat{a}_L^2]$

(b) $[\hat{a}_L, \hat{a}_L\hat{a}_R]$

(c) $[\hat{a}_L\hat{a}_R, \hat{a}_R\hat{a}_L]$

12.143 Prove that $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$ for any linear operators $\hat{A}$, $\hat{B}$, and $\hat{C}$.

12.144 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator. Calculate the next two.

12.145 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator. Verify that the lowering operator $\hat{a}_L$ acting on ψ_2 (the "second excited state") gives you ψ_1.

12.146 It's possible to find the ground state energy of the quantum oscillator using operators rather than calculus.

(a) Rewrite Equation 12.10.3 in terms of $\hat{a}_R\hat{a}_L$ rather than $\hat{a}_L\hat{a}_R$.

(b) Let ψ_0 be the ground state eigenfunction. Don't use the formula we derived; just leave it as ψ_0 and put it into the equation you just wrote. What value does λ have to have in order for this equation to work? *Hint*: remember what $\hat{a}_L$ does when it acts on ψ_0.

12.147 In this problem you will examine the differential equation $\hat{B}\hat{A}\psi + \lambda\psi = 0$ where $\hat{A}$ and $\hat{B}$ are two linear operators about which you know only one thing: $[\hat{A}, \hat{B}] = 5$.

(a) We begin by assuming that we have already found one solution. Write an equation that asserts "ψ_m is an eigenfunction of this differential equation, with eigenvalue λ_m."

(b) Show that $\psi_n = \hat{B}\psi_m$ is an eigenfunction of the same differential equation and find its eigenvalue λ_n in terms of λ_m.

(c) Write a differential equation that you could solve to find the eigenfunction with the lowest eigenvalue. This differential equation will involve the unknown operators so you cannot solve it.

(d) Find the lowest eigenvalue for this equation, and use that to find what all the possible eigenvalues are. (If you're stuck on how to find the lowest eigenvalue you may find it helpful to look at Problem 12.146.)

12.148 Derive Equation 12.10.2 from Equation 12.10.1 using the substitutions given in the Explanation (Section 12.10.2).

12.149 In this problem you will prove that the raising operator works as advertised.

(a) Prove that if ψ_m is an eigenfunction of Equation 12.10.2 with eigenvalue c_m, then $d\psi_m/dy - y\psi_m$ is also an eigenfunction, with eigenvalue $c_m + 2$. Your proof should just involve functions and derivatives, with no operator notation.

(b) Prove that if ψ_m is an eigenfunction of Equation 12.10.3 with eigenvalue c_m, then $\hat{a}_R\psi_m$ is also an eigenfunction, with eigenvalue $c_m + 2$. Your proof should use operators and commutators, with no derivatives written out.

12.150 In the Explanation (Section 12.10.2) we said that the equation $d\psi_0/dy + y\psi_0 = 0$ would give us the ground state of the quantum harmonic oscillator.

(a) Explain how we know that the solution to this particular equation will give us the eigenfunction with the lowest eigenvalue.

(b) Solve the equation by separating variables.

(c) Plug your solution into Equation 12.10.2 to show that it works, and to find the corresponding eigenvalue.

12.151 In the Explanation (Section 12.10.2) we listed the first three eigenfunctions of the quantum oscillator, and we noted that the arbitrary constants in front are determined by the normalization condition $\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$. Use that condition to calculate C_0. You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. (You solved a minor variation of that integral in Section 12.3 Problem 12.49).

12.152 A particle with potential energy $V = (1/4)\kappa x^4$ oscillates, but it is not a "simple harmonic oscillator." Schrödinger's equation for that particle is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{4}\kappa x^4 \psi = E\psi$$

(a) Define a new independent variable y and a new eigenvalue λ and use them to rewrite this without any constants other than λ in it. Your answer should look similar (but not identical) to Equation 12.10.2.

(b) Define $\hat{a}_1 = D + y^2$ and $\hat{a}_2 = D - y^2$ and rewrite the differential equation in terms of these operators, with no explicit derivatives. (See Equation 12.10.3 for example.) There is more than one possible way to do this. *Hint*: begin by calculating $\hat{a}_1\hat{a}_2$ and $\hat{a}_2\hat{a}_1$.

(c) Calculate the commutator $[\hat{a}_1, \hat{a}_2]$.

(d) Assume $\psi_m(y)$ is an eigenfunction with eigenvalue c_m. Plug in $\psi_n = \hat{a}_1\psi_m$ and show that it is *not* an eigenfunction. As part of this process you should rearrange your equation (by using the commutator) so you can make use of the fact that ψ_m is an eigenfunction.

(e) Explain what went wrong. What was it about the commutator $[\hat{a}_1, \hat{a}_2]$ that meant the trick didn't work here the way it did for the simple harmonic oscillator equation?

12.153 A hydrogen atom consists of an electron orbiting about a proton. There is an operator $\hat{L}_z$ that corresponds to the z-component of angular momentum in the following sense. If the particle is in a state where it has a definite z-component of angular momentum μ, then the particle's wavefunction $\psi(x, y, z)$ obeys the eigenvalue equation $\hat{L}_z\psi = \mu\psi$. In other words the eigenfunctions of this equation represent the possible states of definite z-angular momentum. Similar equations hold for the operators $\hat{L}_x$ and $\hat{L}_y$.

It's possible to write these operators out explicitly in terms of x, y, z, $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, but we're not going to bother. All you need to know about them is their commutation relations: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, and $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. To find the eigenvalues of the $\hat{L}_z$ equation, we define the ladder operators $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$.

(a) Calculate $[\hat{L}_z, \hat{L}_+]$ and $[\hat{L}_z, \hat{L}_-]$. Simplify your answers as much as possible.

(b) Suppose ψ_m is an eigenfunction of $\hat{L}_z$ with z-angular momentum $\mu = c_m$. Show that $\hat{L}_+\psi_m$ and $\hat{L}_-\psi_m$ are also eigenfunctions of $\hat{L}_z$, and find their z-angular momenta.