10.9 Reduction of Order and Variation of Parameters

"Reduction of Order" and "Variation of Parameters" are two different formulas that can be used to find solutions to a differential equation based on other known solutions.

10.9.1 Discovery Exercise: Reduction of Order

Consider the equation

$$x^2 y''(x) - (2x^2 + x)y'(x) + (x^2 + x)y(x) = 0 \tag{10.9.1}$$

1. Confirm that $y_1 = e^x$ is a solution to this equation.
2. The second solution is harder to guess, but you can make it easier by writing it in the form $y_2 = u(x)y_1(x)$, where y_1 is the solution we just gave you and $u(x)$ is an unknown function. Plug this into the differential equation to get a differential equation for $u(x)$. Simplify your answer as much as possible.

   See Check Yourself #67 in Appendix L
3. Your equation for u should have u'' and u' in it, but not u by itself. Make the substitution $v = u'$ to get a first-order differential equation for v.
4. Solve the equation you wrote for $v(x)$ and use that to find $u(x)$.
5. Write the general solution to Equation 10.9.1. Plug it in and verify that it works.

The technique you just used is called "reduction of order." When you have one solution y_1 to a linear, second-order ODE , the guess $y_2 = uy_1$ will give you a *first-order* ODE to solve for u'. This is one of two techniques you will learn about in this section.

10.9.2 Explanation: Reduction of Order and Variation of Parameters

Given a linear, inhomogeneous, second-order[8] differential equation, you can solve it if you can do the following three steps (as discussed in Section 10.2).

1. Find two linearly independent functions that solve the complementary homogeneous equation. Here we will call these functions $y_{c1}(x)$ and $y_{c2}(x)$.
2. Find a particular solution to the original inhomogeneous equation. We will refer to this solution as $y_p(x)$.
3. The general solution you are looking for is the sum of the particular and complementary solutions: $y_p(x) + C_1 y_{c1}(x) + C_2 y_{c2}(x)$.

"Reduction of order" helps with Step 1: it assumes you have found one solution to the complementary equation, and it gives you a way to find a second. "Variation of parameters" can take care of Step 2: it assumes you have already found both linearly independent solutions to the homogeneous equation, and it gives you a way to get from those to a particular solution of the inhomogeneous equation.

[8] In this section we are focusing on second-order equations but the methods can be generalized to higher orders.

Reduction of Order

A linear second-order homogeneous differential equation should have two linearly independent solutions. Reduction of Order is a formula that starts with one of these solution and finds the second. If you did the Discovery Exercise (Section 10.9.1) you've used this approach on one particular equation. You start with one solution y_{c1} and write the second one in the form $y_{c2} = u(x)y_{c1}(x)$. When you plug this into the differential equation you get a *first-order* differential equation for u', which you can solve by separation of variables.

In Problem 10.182 you'll apply this technique to a generic second-order linear ODE. The result is the following formula.

Reduction of Order

Given a second-order homogeneous differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, and given one solution $y_{c1}(x)$, a second solution is given by:

$$y_{c2} = uy_{c1} \tag{10.9.2}$$

where u is a function that satisfies the equation:

$$\ln(u') = -\int \left(2\frac{y'_{c1}}{y_{c1}} + a_1\right) dx \tag{10.9.3}$$

These formulas come with the same warning as several other techniques in this chapter: make sure to write your differential equation in the form given above, which includes having no factor in front of $y''(x)$.

Equation 10.9.3 gives you the function $u'(x)$. There is no guarantee that you can integrate that to find the function you need. (Sorry: no technique is perfect.) But if you can, Equation 10.9.2 then gives you the second solution you need to find the general solution.

In the example below we use reduction of order to derive a result that we pulled out of a hat in Section 10.2: the second solution to a Cauchy-Euler equation with only one power solution.

EXAMPLE Reduction of Order

Question: Solve the equation $x^2y'' - 5xy' + 9y = 0$.

Solution:
The decreasing powers of x suggest the guess x^k.

$$y = x^k \quad \rightarrow \quad y' = kx^{k-1} \quad \rightarrow \quad y'' = k(k-1)x^{k-2}$$

Plug this into the differential equation.

$$k(k-1)x^k - 5kx^k + 9x^k = 0 \quad \rightarrow \quad k^2 - 6k + 9 = 0 \quad \rightarrow \quad k = 3$$

So $y = x^3$ is one valid solution. But because the quadratic equation for k had a double root, we don't have a second solution. That's where reduction of order comes in. We begin by writing the differential equation in the proper form—that is, make sure there is no coefficient in front of the y'' term.

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$$

Now we plug $y_{c1} = x^3$ and $a_1 = -5/x$ into Equation 10.9.3.

$$\begin{aligned}
\ln(u') &= -\int \left(2\frac{(3x^2)}{x^3} - \frac{5}{x}\right) dx \\
\ln(u') &= -\int \frac{1}{x}\,dx = -\ln x = \ln\left(x^{-1}\right) \\
u' &= \frac{1}{x} \\
u &= \ln x
\end{aligned}$$

So the second solution to this equation (from Equation 10.9.2) is $y_{c2} = x^3 \ln x$. Note that we did not need arbitrary constants in our integration; we will put them into our general solution.

$$y = Ax^3 + Bx^3 \ln x$$

You can (and should) plug this solution back into the original differential equation and verify that it works.

Variation of Parameters

Variation of Parameters is a formula for finding a particular solution to an inhomogeneous differential equation, based on two (already found) solutions to the complementary homogeneous equation.

Variation of Parameters

Given a second-order differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$, and given two linearly independent solutions y_{c1} and y_{c2} to the complementary homogeneous equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, a particular solution to the inhomogeneous equation is given by:

$$y_p(x) = uy_{c1} + vy_{c2} \tag{10.9.4}$$

where u and v are functions that satisfy the equations:

$$\begin{aligned}
u' &= \frac{y_{c2}}{y'_{c1}y_{c2} - y_{c1}y'_{c2}} f(x) \\
v' &= \frac{-y_{c1}}{y'_{c1}y_{c2} - y_{c1}y'_{c2}} f(x)
\end{aligned} \tag{10.9.5}$$

These formulas come with the same warning as several other techniques in this chapter: make sure to write your differential equation in the form given above, which includes having no factor in front of $y''(x)$.

Equations 10.9.5 give you the functions $u'(x)$ and $v'(x)$. There is no guarantee that you can integrate those to find the functions you need. (Sound familiar?) But if you can, Equation 10.9.4 then gives you a particular solution, which you combine with your already-found complementary solutions to find the general solution.

Below we will give an example demonstrating the process, and then show where the formula comes from.

EXAMPLE Variation of Parameters

Question: Solve the equation $x^2\left(d^2y/dx^2\right) - 2y = \ln x$.

Solution:

We start by rewriting this in the correct form, which means dividing by x^2 so the coefficient of $y''(x)$ is one. That leaves $y''(x) - (2/x^2)y = (\ln x)/x^2$, so $f(x) = (\ln x)/x^2$.

Next we solve the complementary homogeneous equation $y''(x) - (2/x^2)y = 0$, which we will attack by plugging in the guess $y = x^k$.

$$k(k-1)x^k - 2x^k = 0 \quad \rightarrow \quad (k^2 - k - 2)x^k = 0 \quad \rightarrow \quad k = -1 \text{ or } k = 2$$

So our two homogeneous solutions are:

$$y_{c1} = \frac{1}{x} \qquad \textit{and} \qquad y_{c2} = x^2$$

These are the solutions to the complementary homogeneous equation, and we will need them when we write our general solution. But now we are going to plug them into Equations 10.9.5 to find the new functions u and v that we need for our *particular* solution.

$$u' = \frac{x^2}{-(1/x^2)x^2 - (1/x)(2x)}\frac{\ln x}{x^2} = -\frac{1}{3}\ln x$$
$$v' = -\frac{1/x}{-(1/x^2)x^2 - (1/x)(2x)}\frac{\ln x}{x^2} = \frac{1}{3}\frac{\ln x}{x^3}$$

These can be integrated by parts to give

$$u = \frac{1}{3}x - \frac{1}{3}x\ln x$$
$$v = -\frac{1}{12x^2} - \frac{1}{6x^2}\ln x$$

(You do not need to include $+C$ because we are not looking for a general solution; any particular solution will do.) Equation 10.9.4 tells us how to put those together to find a particular solution to our equation: $y_p(x) = uy_{c1} + vy_{c2}$. After a bit of algebra, this gives

$$y_p(x) = \frac{1}{4} - \frac{1}{2}\ln x$$

Remember that the general solution is a sum of the particular and homogeneous solutions!

$$y(x) = \frac{1}{4} - \frac{1}{2}\ln x + \frac{C_1}{x} + C_2x^2$$

We leave it to you to confirm that this solves the original equation. (Go on, it's good for you.)

Where Did That "Variation of Parameters" Formula Come From?

A linear second-order differential equation can be written in the form:

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{10.9.6}$$

You might think that we're going to start there and derive Equation 10.9.4, but that isn't the plan: we're going to treat Equation 10.9.4 as a guess, plug it in, and end up at Equation 10.9.5. As always, the guess justifies itself when it works.

So we begin with our guess and take some derivatives. (This involves a few product rules, but also some clever grouping.) Remember as we go that we are starting with y_{c1} and y_{c2} (the already-found solutions to the complementary homogeneous equation) and looking for u and v.

$$\begin{aligned}
y_p(x) &= uy_{c1} + vy_{c2} \\
y_p'(x) &= u'y_{c1} + uy_{c1}' + v'y_{c2} + vy_{c2}' \\
&= uy_{c1}' + vy_{c2}' + (u'y_{c1} + v'y_{c2}) \\
y_p''(x) &= u'y_{c1}' + uy_{c1}'' + v'y_{c2}' + vy_{c2}'' + (u'y_{c1} + v'y_{c2})' \\
&= uy_{c1}'' + vy_{c2}'' + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})'
\end{aligned}$$

Plugging all that into Equation 10.9.6,

$$\begin{aligned}
uy_{c1}'' + vy_{c2}'' + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})' + a_1[uy_{c1}' + vy_{c2}' + (u'y_{c1} + v'y_{c2})] \\
+ a_0(uy_{c1} + vy_{c2}) = f(x)
\end{aligned}$$

Rearranging:

$$\begin{aligned}
u(y_{c1}'' + a_1y_{c1}' + a_0y_{c1}) + v(y_{c2}'' + a_1y_{c2}' + a_0y_{c2}) + a_1(u'y_{c1} + v'y_{c2}) + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} \\
+ v'y_{c2})' = f(x)
\end{aligned}$$

Now comes the good part. y_{c1} is a solution to the complementary homogeneous equation, which means by definition that $y_{c1}'' + a_1y_{c1}' + a_0y_{c1} = 0$. Similarly of course for y_{c2}. So both of the first terms go away, leaving this.

$$a_1(u'y_{c1} + v'y_{c2}) + (u'y_{c1}' + v'y_{c2}') + (u'y_{c1} + v'y_{c2})' = f(x) \tag{10.9.7}$$

You might think we're going to keep manipulating until we end up with u=<*something*> but that is not possible, because there is not just one unique solution. And we don't need one: we only need *one particular* solution, or in other words, anything that works. The easiest way to make Equation 10.9.7 work is to choose u and v such that:

$$\begin{aligned}
u'y_{c1} + v'y_{c2} &= 0 \\
u'y_{c1}' + v'y_{c2}' &= f(x)
\end{aligned}$$

These are now algebra equations, which you can easily solve for u' and v' to arrive at Equation 10.9.5.

10.9.3 Problems: Reduction of Order and Variation of Parameters

10.177 **Walk-Through: Reduction of Order.** In this problem you will solve the equation $x^4y'' + (x + x^3)y' - (1 + x^2)y = 0$.

(a) Show that $y_{c1} = x$ is one valid solution.

(b) Rewrite this equation in the form $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$. Identify the functions $a_1(x)$ and $a_0(x)$.

(c) Use Equation 10.9.3 to find the function $u(x)$.

(d) Use Equation 10.9.2 to find the second solution $y_{c2}(x)$ to this equation.

(e) Verify that your $y_{c2}(x)$ is a valid solution to this differential equation.

(f) Write the general solution to this differential equation.

10.178 In Section 10.2 we discussed the equation $y'' + 6y' + 9y = 0$. The guess $y = e^{kx}$ leads to one solution, $y = e^{-3x}$. We then suggested trying $y = xe^{-3x}$, but gave no indication of where this second solution came from. Use reduction of order to find this second solution for yourself.

10.179 In Section 10.2 we discussed the equation $x^2y'' + 5xy' + 4y = 0$. The guess $y = x^k$ leads to one solution, $y = 1/x^2$. We then suggested trying $y = (\ln x)/x^2$, but gave no indication of where this second solution came from. Use reduction of order to find this second solution for yourself.

10.180 One solution to the equation $xy'' + (3 - 2x)y' + (x - 3)y = 0$ is $y = e^x$. Find a second solution that is linearly independent of the first.

10.181 Find the general solution to the equation $x^3y'' + (x^2 + x)y' - (x + 1)y = 0$. You will need to begin by playing around until you find one simple solution that works. Reduction of order will then give you the second solution. The last step of finding $u(x)$ involves a tricky integral, but you can evaluate it by parts.

10.182 In this problem you will derive the formula for reduction of order. Remember the scenario: an existing solution $y_{c1}(x)$ has been found, and you are now looking for a second solution by guessing $y_{c2}(x) = u(x)y_{c1}(x)$. The goal is to find the unknown function $u(x)$.

(a) Begin by plugging the function $y_{c2} = uy_{c1}$ into the differential equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$. (This will involve a few product rules.) Write your answer in the form

$$u''\langle\text{something}\rangle + u'\langle\text{something}\rangle + u\langle\text{something}\rangle = 0$$

(b) Explain why the u<something> must be zero, and can therefore be ignored. *Hint*: remember what y_{c1} stands for.

(c) The resulting differential equation contains u' and u'' but no u. As we saw in Section 10.8 this suggests the substitution $v = u'$. Write the resulting first-order differential equation for $v(x)$. (This is where "reduction of order" gets its name.)

(d) The resulting differential equation is separable. Solve for $v(x)$. If all goes well, you should end up deriving Equation 10.9.3.

10.183 Consider the differential equation $y'''(x) = y(x)$.

(a) This equation has one relatively obvious solution (beside $y(x) = 0$). Write it down and call it $y_{c1}(x)$.

(b) To find the other two solutions, begin by substituting $y_2(x) = u(x)y_{c1}(x)$ into the differential equation, using the $y_{c1}(x)$ you found in Part (a). Write the resulting third-order differential equation for $u(x)$.

(c) Convert the third-order equation for u into a second-order equation for $v = u'$ and solve it. Then use your solution to find u.

(d) Write the general solution to $y'''(x) = y(x)$.

10.184 **Walk-Through: Variation of Parameters.** In this problem you will solve the equation $(\sin x)\left[y''(x) + y(x)\right] = 1$.

(a) Rewrite this equation in the form $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$. Identify the functions $a_1(x)$, $a_0(x)$, and $f(x)$.

(b) Write the complementary homogeneous equation. Find two linearly independent solutions. (We hope you can see them both just by looking.)

(c) Calling the two functions you wrote in Part (b) $y_{c1}(x)$ and $y_{c2}(x)$, plug them into Equation 10.9.5 so you have two equations for the functions $u'(x)$ and $v'(x)$.

(d) Integrate your answers to find $u(x)$ and $v(x)$. You should be able to do both integrals by hand.

(e) Use Equation 10.9.4 to put your solutions together into a particular solution.

(f) Put your answers together to write the general solution to the original differential equation. Your solution should include two arbitrary constants C_1 and C_2.

(g) Verify that your solution works.

In Problems 10.185–10.187 solve the given equation by variation of parameters. You may find it helpful to first work through Problem 10.184 as a model.

10.185 $y'' - 10y' + 25y = e^{5x}/x^2$

10.186 $y'' - 4y' = 8e^{kx}$ (where k is a constant)

10.187 $2y'' - 5y' + 2y = e^{4x}$

Problems 10.188–10.192 have four parts each.

- Write the complementary homogeneous equation. In some cases we will give you a solution y_{c1} to this equation; if we don't, play around a bit until you find one.
- Use reduction of order to find the other solution y_{c2} to the complementary homogeneous equation.
- Use variation of parameters to find a particular solution y_p to the original equation.
- Write the general solution to the original equation.

10.188 $y''(x) + (3\tan x)y'(x) - 2y(x) = \cos^4 x$. Begin with $y_{c1} = \sin x$.

10.189 $(x-1)y''(x) - xy'(x) + y(x) = (x-1)^2$. Begin with $y_{c1} = e^x$.

10.190 $xy''(x) - (2x+1)y'(x) + (x+1)y(x) = x^2$

10.191 $y''(x) + (\tan x - (2/x))\,y'(x) - \left(\tan(x)/x - (2/x^2)\right) y(x) = x\cos x$. Begin with $y_{c1} = x$.

10.192 $y''(x) - (2\cot x + 1)y'(x) + (1 + \cot x + 2\cot^2 x)y(x) = \sin x$. Begin with $y_{c1} = \sin x$.

10.193 Variation of parameters tells you the derivatives $u'(x)$ and $v'(x)$, but you have to integrate to find the functions u and v. Since those are indefinite integrals, they should normally include arbitrary constants. Recall, however, that the general solution you get is $y(x) = uy_{c1} + vy_{c2} + C_1y_{c1} + C_2y_{c2}$, where C_1 and C_2 are arbitrary constants. Explain why, if you add arbitrary constants to u and v, it doesn't change this solution.

10.194 In this section we have focused on second-order differential equations, but variation of parameters can be used for linear equations of any order. In this problem you will derive and use the formula for the first-order equation:

$$\frac{dy}{dx} + a_0 y = f(x) \tag{10.9.8}$$

You will be looking for a solution of the form $y_p = u(x)y_c(x)$ where u is an unknown function and y_c is a solution to the complementary homogeneous equation.

(a) Find y_p' by the product rule. Then plug y_p and y_p' into Equation 10.9.8.

(b) Collect the terms that have u in them, and factor out the u.

(c) The resulting terms in parentheses add up to zero, and can be dropped. Why?

(d) Solve the remaining equation for $u'(x)$. This is the equation you have been looking for.

(e) Use your formula to solve the following equation.

$$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{1}{x^4+5}$$

(You will begin by solving the complementary homogeneous equation by separation of variables.)

10.195 A 1 kg block is attached to a spring with spring constant 12 N/m and damping force $F_d = -bv$, $b = 7$ N·s/m. The block is acted on by an external force F_e.

(a) Write the differential equation for the position of the block, $x(t)$.

(b) Find the complementary solutions to this equation.

(c) Use variation of parameters to find a particular solution. Your answer will include integrals of the unknown function $F_e(t)$.

(d) Using your result from Part (c), find the particular solution $x_p(t)$ for each of the following driving forces.

i. $F_e = 5$ N
ii. $F_e = ae^{t/\tau}$, with $a = 2$ N and $\tau = 2$ s
iii. $F_e = ate^{t/\tau}$ with $a = 3$ N/s and $\tau = 2$ s

10.196 The charge on the capacitor in an RLC circuit obeys the equation $Q''(t) + 6Q'(t) + 5Q(t) = V'(t)$, where $V(t)$ is the voltage at the voltage source.

(a) Find the complementary solutions to this equation.

(b) Use variation of parameters to find a particular solution. Your answer will include integrals involving the unknown function $V'(t)$.

(c) Using your result from Part (c), find the particular solution $Q_p(t)$ for each of the following driving forces. (Be careful to use $V'(t)$ and not $V(t)$ in your formulas.)

i. $V = 5$
ii. $V = e^{2t}$
iii. $V = te^t$

10.197 In Chapter 11 we will solve for the motion of a string of length L held fixed at both ends and subjected to a uniform, time-dependent external driving force. We will reduce that problem to solving the following differential equation in which $b_{yn}(t)$ is the function we are solving for, $a(t)$ represents the time dependence of the external force, and n, L, and v are constants.[16]

$$-\frac{n^2\pi^2}{L^2} b_{yn}(t) - \frac{1}{v^2}\frac{d^2 b_{yn}(t)}{dt^2} = \frac{4}{n\pi} a(t)$$

(a) Find the complementary solutions to this ODE for $b_{yn}(t)$.

(b) Use variation of parameters to find a particular solution. Your answer will include integrals with the unknown function $a(t)$ in them.

(c) Find a particular solution $b_{yn}(t)$ for $a(t) = k$ (a constant external force).

(d) Find a particular solution $b_{yn}(t)$ for $a(t) = t$ (a linearly increasing external force).

10.198 *[This problem depends on Problem 10.197.]* In Problem 10.197 you found the Fourier coefficients for the motion of a vibrating string subject to an external driving force $a(t)$. In this problem, use a computer to find a particular solution for $b_{yn}(t)$ for the external force law: $a(t) = \sin t$.

[16] Some physical background you don't need for this problem: $b_{yn}(t)$ is the nth Fourier coefficient of the shape of the string $y(x)$ at time t, for odd n only. For even n the Fourier coefficients obey the same equation but with 0 on the right-hand side.