10.6 Linearly Independent Solutions and the Wronskian

The "Wronskian" is a tool that can help you determine if you have found linearly independent solutions to a differential equation. That, in turn, helps you know when you have found the general solution.

10.6.1 Discovery Exercise: Linearly Independent Solutions and the Wronskian

1. Each of the following functions is a valid solution to the differential equation $y''(x) + k^2 y(x) = 0$. But four of these functions represent (in different forms) the *general solution,* and the other three do not. Which ones?

(a) $y(x) = \sin(kx)$
(b) $y(x) = A\sin(kx) + B\cos(kx)$
(c) $y(x) = A\sin(kx + \phi)$
(d) $y(x) = e^{ikx}$
(e) $y(x) = Ae^{ikx} + B\sin(kx)$
(f) $y(x) = Ae^{ikx} + Be^{-ikx}$
(g) $y(x) = Ae^{ikx+\phi}$

See Check Yourself #66 in Appendix L

2. Each of the following functions is a valid solution to the differential equation $4x^2 y''(x) + y(x) = 0$. But two of these functions represent (in different forms) the *general solution,* and the other three do not. Which ones?

(a) $y(x) = \sqrt{x}$
(b) $y(x) = \sqrt{x}\ln x$
(c) $y(x) = A\sqrt{x} + B\sqrt{x}\ln x$
(d) $y(x) = A\sqrt{x} + B\sqrt{x}(1 + \ln x)$
(e) $y(x) = (A + B)\sqrt{x}\ln x$

10.6.2 Explanation: Linearly Independent Solutions and the Wronskian

The differential equation $y'' + k^2 y = 0$ has two solutions: $y_1 = \sin(kx)$ and $y_2 = \cos(kx)$. Those are both valid individual solutions, so superposition says we can combine them to form another solution.

$$y = Ay_1(x) + By_2(x) = A\sin(kx) + B\cos(kx) \tag{10.6.1}$$

Equation 10.6.1 is not just a solution; it is the general solution. All solutions can be written as forms of Equation 10.6.1 with the proper choices of the constants A and B.

Let's play that discussion back with a slight change. The differential equation $y'' + k^2 y = 0$ has two solutions: $y_1 = 3\sin(kx)$ and $y_2 = -5\sin(kx)$. Those are both valid individual solutions, so superposition says we can combine them to form another solution.

$$y = Ay_1(x) + By_2(x) = 3A\sin(kx) - 5B\sin(kx) \tag{10.6.2}$$

Equation 10.6.2 is a valid solution, but it is *not* the general solution. It's just an obfuscated way of writing $A\sin(kx)$. Many valid solutions, such as $y = 2\cos(kx)$, cannot fit into this form.

The second solution is not general because its two solutions y_1 and y_2 are *linearly dependent*. They are of course not the same function, but one is a constant multiple of the other.

Definition and Use: Two Linearly Dependent Functions

Two functions $y_1(x)$ and $y_2(x)$ are "linearly dependent" if and only if $y_1 = ky_2$ for some constant k.[3]

If $y_1(x)$ and $y_2(x)$ are solutions to a linear second-order homogeneous differential equation, then $Ay_1(x) + By_2(x)$ is also a solution. But it is the *general* solution if and only if y_1 and y_2 are linearly independent functions.

In our examples above it was obvious which functions were linearly dependent. In other cases it can be harder to tell, but there is a general test you can apply.

Definition and Use: The Wronskian

The "Wronskian" of two functions $y_1(x)$ and $y_2(x)$ is:

$$W(x) = y_1 y_2' - y_1' y_2$$

Given a linear, second-order, homogeneous differential equation $y''(x) + a_1 y'(x) + a_0 y(x) = 0$ where a_1 and a_0 are continuous functions on an open interval I (this interval might be "all real numbers" but it doesn't have to be), and given two functions $y_1(x)$ and $y_2(x)$ that solve this equation on I…

- The Wronskian $W(x)$ of y_1 and y_2 is zero everywhere in I, or it is non-zero everywhere in I. (It cannot be zero for some x-values and non-zero for others.)
- If $W(x) = 0$ then the two solutions are linearly dependent.
- If $W(x) \neq 0$ then the two solutions are linearly independent, and $Ay_1(x) + By_2(x)$ is therefore the general solution.

EXAMPLE The Wronskian

In Chapter 12 we will show that the linear second-order homogeneous differential equation $(1 - x^2)y'' - 2xy' + l(l+1)y = 0$ on the open interval $-1 < x < 1$ has the following two solutions.

$$y_1 = 1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{6!}x^6 + \ldots$$
$$y_2 = x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{7!}x^7 + \ldots$$

The function $Ay_1(x) + By_2(x)$ represents the general solution if and only if y_1 and y_2 are linearly independent—that is, if their Wronskian is non-zero. We begin by taking derivatives.

$$y_1' = -l(l+1)x + \frac{l(l+1)(l-2)(l+3)}{3!}x^3 - \frac{l(l+1)(l-2)(l+3)(l-4)(l+5)}{5!}x^5 + \ldots$$
$$y_2' = 1 - \frac{(l-1)(l+2)}{2!}x^2 + \frac{(l-1)(l+2)(l-3)(l+4)}{4!}x^4 - \frac{(l-1)(l+2)(l-3)(l+4)(l-5)(l+6)}{6!}x^6 + \ldots$$

[3] Technically the function $f(x) = 0$ is linearly dependent with all other functions. Our definition therefore leaves out the case $y_2(x) = 0$ and $y_1(x) \neq 0$. But we are looking here for non-trivial solutions to homogeneous differential equations, so we're going to ignore the zero function for the rest of this section.

The Wronskian is $W(x) = y_1 y_2' - y_1' y_2$. Multiplying every term in y_1 by every term in y_2' does not look practical, but remember that we don't need to prove that $W(x) = 0$ for all x-values; any x-value in our interval can serve as a representative for the entire interval. And it's easy to check at $x = 0$. The first term of $y_1 y_2'$ is 1. Every other term of $y_1 y_2'$, and *every* term of $y_1' y_2$, will go to zero at $x = 0$. We conclude that $W(0) = 1$ which means y_1 and y_2 are linearly independent, so we have the general solution.

We hope the example above demonstrates that the Wronskian is both easy and powerful. But its use rests on the three claims we made in the box "Definition and Use: The Wronskian" and none of these three claims is obvious. We are going to establish the connection between the Wronskian and linear dependence in two ways. The explanation below presents a coherent way to think about what the Wronskian represents and why it demonstrates linear dependence. That conceptual discussion will then allow us to figure out how to generalize the Wronskian to more than two functions. In Problems 10.116–10.117 you will prove two of these claims in a different way: more direct, but possibly less useful in the long run.

Why are Those Three Assertions True?

In justifying the claims we made about the Wronskian, it's easiest to start with the last one. If two functions y_1 and y_2 are linearly dependent then $y_1 = ky_2$. That in turn means $y_1' = ky_2'$, and you can easily plug these in to the definition of the Wronskian and conclude that $W = 0$. Thus if $W \neq 0$, the two functions are linearly independent.

The second claim is the converse of the third: if $W = 0$ everywhere in the interval I then y_1 and y_2 must be linearly dependent in that interval. First let's rewrite the assertion that $W = 0$ at some point $x = a$.[4]

$$W(a) = 0 \quad \Leftrightarrow \quad y_1(a)y_2'(a) = y_2'(a)y_1(a) \quad \Leftrightarrow \quad \frac{y_1(a)}{y_2(a)} = \frac{y_1'(a)}{y_2'(a)} \tag{10.6.3}$$

At the point $x = a$, $y_1 = ky_2$ for some k. (That's always true unless $y_2(a) = 0$.) Equation 10.6.3 tells us that if the Wronskian is zero, the slope of y_1 is also k times the slope of y_2. And now we arrive at the heart of the argument:

If y_1 starts out k times higher than y_2 and increases k times faster than y_2, then it will stay k times higher than y_2.

Of course that's only true for a moment. But remember the first claim: if $W = 0$ anywhere in interval I, then $W = 0$ everywhere in the interval. So as the function climbs with y_1'/y_2' remaining always the same as y_1/y_2, the functions will retain the relationship $y_1 = ky_2$ (as in Figure 10.4), which is what we're trying to show. You can make this particular argument more rigorous by treating the curve as a succession of line segments, and then taking a limit; you will reach this same conclusion in a slightly different way in Problem 10.116.

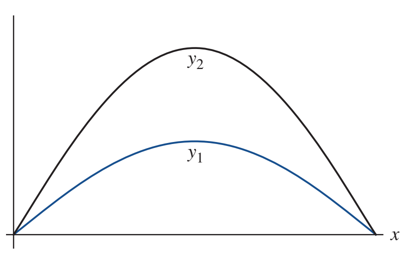

FIGURE 10.4 Two linearly dependent functions. If $y_2 = 2y_1$ then at each point y_2 must be rising or falling twice as fast as y_1.

[4] Our algebra is not valid if $y_2 = 0$ or $y_2' = 0$, but it's not hard to show that the basic conclusions still hold in those cases.

The two arguments we've made so far apply to any two smooth functions[5] y_1 and y_2. If they are linearly dependent then $W = 0$, and if $W = 0$ throughout some interval I then the functions are linearly dependent in that interval. What remains is the first claim: $W = 0$ either everywhere or nowhere in I. That claim is not guaranteed for two arbitrary functions, but it is guaranteed if those functions are both solutions to the same linear second-order differential equation.

Why? We said above y_1 will continue to be proportional to y_2 if y_1' continues to be proportional to y_2'. By the same logic, y_1' and y_2' will continue to be proportional if y_1'' and y_2'' are proportional. (For example, the functions e^x and $x + 1$ have the same value and first derivative at $x = 0$, but they diverge because their second derivatives are different.)

This is where the differential equation comes in. A second-order linear differential equation can be written $y'' = -a_1(x)y' - a_0(x)y$. If you know y and y' at a point, then the differential equation tells you y''. If y_1 and y_2 are both solutions to the same such equation and $y_1/y_2 = y_1'/y_2' = k$, then y_1''/y_2'' must also equal k. By differentiating both sides of the differential equation you can get an expression for y''' and similarly conclude that $y_1'''/y_2''' = k$, and so on up to arbitrarily high orders. Since all the higher order derivatives are proportional, y_1 will continue to grow k times faster than y_2, and W will continue to be zero.

Problem 10.117 will present a more algebraic and more rigorous proof of the first claim. In our opinion, however, this hand-waving argument gives more useful intuition for the Wronskian than the algebraic proof, which is why we put this one in the explanation and that one in the problems.

The Wronskian as a Determinant, or, What if There are More than Two Functions?

If you remember how to take the determinant of a 2×2 matrix then it's easy to see that the following determinant is the Wronskian of y_1 and y_2.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad (10.6.4)$$

Sometimes determinants appear as helpful mnemonics, such as with the cross product or curl—useful, but coincidental—but this is far more than that. Every column in a matrix is a vector. If the determinant of the matrix is zero, then these vectors are linearly dependent. That is one of the primary purposes of the determinant, and look what it means here.

$$W = 0 \quad \Longleftrightarrow \quad \langle y_1, y_1' \rangle = k\langle y_2, y_2' \rangle \quad \Longleftrightarrow \quad y_1 = ky_2,\ y_1' = ky_2'$$

We have already seen that, because y_1 and y_2 are solutions to the same second-order linear ODE, those two relationships are enough to guarantee that the functions are linearly dependent. The payoff for this approach is that it generalizes seamlessly to higher levels. A third-order equation requires three linearly independent functions, a fourth-order requires four, and so on. So we need a general method for determining if n functions are linearly independent—provided, once again, that they are all solutions to the same nth-order linear differential equation.

What does that even mean? If two vectors $\vec{A}$ and $\vec{B}$ are linearly dependent, then $\vec{A} = k\vec{B}$ for some scalar k. For three or more vectors linear dependence is subtler: if $2\vec{A} - 3\vec{B} = 10\vec{C}$ then the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are linearly dependent even if no two of them are.

The same relationship holds for functions. Consider the functions e^{6x}, $\sin x$, and $2e^{6x} - 3\sin x$. Any two of these functions are linearly independent, but the set of three functions is not. If all three of them solved the same third-order linear homogeneous ODE, you could not combine them to form the general solution.

[5] If they aren't differentiable then W isn't defined.

So it can become a subtle business to look at three (or more!) functions and determine if they are linearly dependent. But the Wronskian generalizes to any level because determinants themselves generalize to any level. Consider three functions $y_1(x)$, $y_2(x)$, and $y_3(x)$. The Wronskian of these three functions is given by the following determinant.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix}$$

If the Wronskian is zero then the vectors $<y_1, y_1', y_1''>$, $<y_2, y_2', y_2''>$and $<y_3, y_3', y_3''>$ are linearly dependent. If we also know that these three functions are solutions to the same third-order linear homogeneous differential equation, then their higher order derivatives are all linear functions of these three numbers. So the linear dependence of these three variables is sufficient to conclude that the functions are linearly dependent.

10.6.3 Problems: Linearly Independent Solutions and the Wronskian

In Problems 10.108–10.114 you will be given a linear differential equation and a set of solutions valid on the interval $(-\infty, \infty)$. (You won't actually use the differential equation, but we include it to emphasize that this technique only applies to functions that are solutions to a common ODE.) Use the Wronskian to determine if the functions are linearly independent or dependent.

10.108 $y'' = -k^2y$, $y_1(x) = \sin(kx)$, $y_2 = \cos(kx)$

10.109 $y'' = -k^2y$, $y_1(x) = 3\sin(kx)$, $y_2 = -5\sin(kx)$

10.110 $y'' = -k^2y$, $y_1(x) = \sin(kx)$, $y_2 = e^{ikx}$

10.111 $y'' = -k^2y$, $y_1(x) = \sin(kx)$, $y_2 = \cos(kx)$, $y_3 = e^{ikx}$

10.112 $y''' - 6y'' + 11y' - 6 = 0$, $y_1(x) = 4e^x - 2e^{2x} + e^{3x}$, $y_2(x) = -e^x + 2e^{2x} - e^{3x}$, $y_3(x) = 5e^x + 2e^{2x} - e^{3x}$

10.113 $y'' - 2xy' + 2ky = 0$,

$$y_1(x) = 1 - \frac{2k}{2!}x^2 + \frac{2^2k(k-2)}{4!}x^4 - \frac{2^3k(k-2)(k-4)}{6!}x^6 + \dots,$$

$$y_2(x) = x - \frac{2(k-1)}{3!}x^3 + \frac{2^2(k-1)(k-3)}{5!}x^5 - \frac{2^3(k-1)(k-3)(k-5)}{7!}x^7 + \dots$$

10.114 $y_1(x) = x - 2x^2 - (1/3)x^3 + \dots$, $y_2(x) = 3 + 3x - 33x^2 + 35x^3 + \dots$, $y_3(x) = 3 - 27x^2 + 36x^3 + \dots$ (For this one we are not giving an ODE or a pattern for the rest of the series. Just assume you have three series solutions that start like this.)

10.115 Let $f(x) = 2e^x$ and $g(x) = \sin x + \cos x$ on the interval $[-\pi, \pi]$.

(a) Calculate the Wronskian for these two functions.

(b) Evaluate the Wronskian at $x = 0$.

(c) Evaluate the Wronskian at $x = \pi/2$.

(d) Explain why your answers don't contradict what we've said about the properties of the Wronskian.

(e) Are these two functions linearly dependent on this interval?

10.116 In this problem you will prove the second of our claims about the Wronskian: if the Wronskian of two functions is zero throughout an interval I then the two functions must be linearly dependent.

(a) We defined two linearly dependent functions by the equation $y_1 = ky_2$ for some constant k, which we can also write as:

$$\frac{y_1(x)}{y_2(x)} = k$$

Take the derivative with respect to x of both sides of that equation.

(b) Based on your answer to Part (a), argue that if $W = 0$ then y_1 and y_2 must be linearly dependent.

(c) Why would your argument above not work if $W = 0$ only at a point, and not within a non-zero interval? If you're stuck you may find it helpful to look at our discussion of the functions e^x and $1 + x$ in the Explanation (Section 10.6.2).

10.117 In this problem you will prove the first of our three claims about the Wronskian. If two functions $y_1(x)$ and $y_2(x)$ are both solutions to the same equation $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$ on an open interval I then the Wronskian $W = y_1y_2' - y_1'y_2$ is zero everywhere in I, or it is non-zero everywhere in I. The strategy will be to write and solve an ODE for $W(x)$.

(a) Calculate W'. Your answer should be in terms of y_1, y_2, y_1'', and y_2''. Simplify your answer as much as possible.

(b) Since you know y_1 is a solution to the ODE you can rewrite y_1'' in terms of y_1 and y_1'. Use this substitution and the corresponding one for y_2'' to write W' in terms of y_1, y_2, y_1', and y_2'. Simplify your answer as much as possible.

(c) Replace $y_1y_2' - y_1'y_2$ with W in your expression for W'. The result should only depend on W and a_1.

(d) You just wrote a linear first-order differential equation for $W(x)$. Solve this equation by separating variables to find a formula for $W(x)$ in terms of the unknown function $a_1(x)$. Include an arbitrary constant in front of your solution.

(e) Explain how you can tell by looking at your solution that $W(x)$ is never zero unless $W(x)$ is always zero.

10.118 In this problem you will fill in some of the missing algebra in our discussion of the Wronskian. Consider two functions $y_1(x)$ and $y_2(x)$ that solve the following equation.

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \qquad (10.6.5)$$

Suppose that at $x = c$ we know two facts: $y_1(c) = ky_2(c)$ and $y_1'(c) = ky_2'(c)$. (In the explanation we called the point $x = a$ but we don't want to confuse that a with the coefficient functions in the ODE.)

(a) Show that $y_1''(c) = ky_2''(c)$. *Hint*: This is not guaranteed without the differential equation!

(b) Take the derivative of both sides of Equation 10.6.5.

(c) Show that $y_1'''(c) = ky_2'''(c)$.

Using similar logic you can easily show that $y_1''''(c) = ky_2''''(c)$ and so on for all higher derivatives.

10.119 Consider two functions $f(x)$ and $g(x)$ such that $f(0) = 3g(0)$.

(a) First, suppose both functions are lines. What must be the relationship between their first derivatives (slopes) if $f(x)$ is to continue being three times $g(x)$ as they move to the right?

(b) If $f(x)$ and $g(x)$ are not necessarily lines, then the relationship between $f'(0)$ and $g'(0)$ that you wrote in Part (a) is not enough to show that $f(x) = 3g(x)$. Why not?

(c) If $f(x) = 3g(x)$ for all x-values, then what must be the relationship between $f''(0)$ and $g''(0)$?