CHAPTER 10

Methods of Solving Ordinary Differential Equations (Online)

10.3 Phase Portraits

Just as a slope field (Section 1.4) gives us a way to visualize the solutions to a first-order ODE, a phase portrait is a way of visualizing the solutions to two (or more) coupled first-order ODEs, or to a single second-order ODE.

10.3.1 Explanation: Phase Portraits

In Section 1.7 (see felderbooks.com) we introduced "coupled" differential equations. Such equations occur when two variables depend on each other. For instance, dx/dt may depend on both x and y, and dy/dt may also depend on x and y. A solution would be a pair of equations $x(t)$ and $y(t)$ that solve both equations simultaneously.

Our first example was the math problem of Romeo and Juliet.[2] Romeo's love grows the more Juliet loves him ($dR/dt = J$), while Juliet's love diminishes the more Romeo loves her ($dJ/dt = -R$). The state of the system at any moment is the value of the two functions R and J. If you know those two numbers at any moment in time you can figure out how they will evolve for all future times.

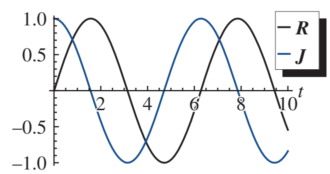


FIGURE 10.1 Romeo and Juliet's feelings for each other oscillate out of phase.

In Chapter 1 we solved these equations and found that R and J oscillate sinusoidally, 90° out of phase, as indicated in Figure 10.1. In this section we will arrive at the same conclusion by a graphical method. Like slope fields, the method of "phase portraits" allows us to visualize the possible behaviors of a system, in this case of two coupled ODEs. Although we're introducing them in the context of this simple problem that we can solve exactly, phase portraits can be used to understand the behavior of systems whose equations can't be solved analytically.

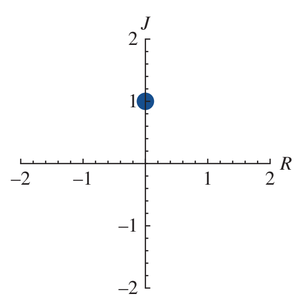


FIGURE 10.2 Juliet loves Romeo.

The phase portrait for the Romeo and Juliet system is a plot with R on one axis and J on another. For instance, Figure 10.2 shows that Juliet loves an indifferent Romeo. Every point in this space represents one possible state of the Romeo-and-Juliet system.

Figure 10.2 is not like most graphs you have worked with. You are accustomed to the variable on the y-axis depending on

[2]adapted with permission from *Nonlinear Dynamics and Chaos* by Steven Strogatz.

the variable on the x-axis. In this case both the x- and y-axes represent dependent variables. The independent variable, time, does not appear in the diagram at all. So the state $R = 0$, $J = 1$ shown on the plot could be an initial condition ($t = 0$) or it could occur at any other time. We can't say.

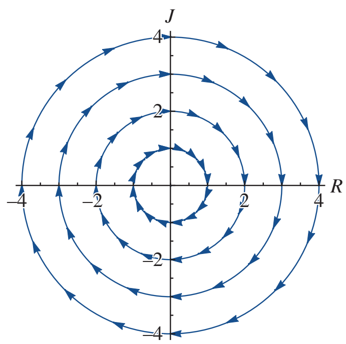


FIGURE 10.3 A phase portrait showing possible solutions to the Romeo and Juliet equations.

What we *can* say, based on the differential equations, is how this state will evolve. $dR/dt = J$ tells us that R will increase; $dJ/dR = -R$ tells us that J will hold steady. So if the system is ever momentarily in the state shown in Figure 10.2, its next shift will be to the right. From there the positive R will start causing J to decrease. If you follow this logical progression you will end up describing a circle back around to the point $(0, 1)$ where we started, and so on forever. This circle describes the same progression, and for the same reasons, that Figure 10.1 described.

But that circle is only the particular solution that starts at the point $(0, 1)$. If the system starts farther from the origin it will trace out a similar circle with a larger radius. We can therefore represent the system by Figure 10.3, a "phase portrait." Just like Figure 10.1, this new representation shows Romeo and Juliet's loves oscillating 90° out of phase with each other in a perpetual cycle of love and hate. Each of these representations has an advantage relative to the other. The advantage of Figure 10.1 is that it includes the time, which Figure 10.3 does not. Looking at the phase portrait we can see that R and J will oscillate, but there's no way to know how long each oscillation takes, or even whether it's going faster at some points of the cycle than at others. The advantage of the phase portrait is that it shows not just a single trajectory, but a whole family of possible trajectories corresponding to different initial conditions. When we look at a phase portrait that includes enough trajectories we can see in one plot all the possible behaviors of the system.

Definition: Phase Portrait

A "phase portrait" is a plot showing the possible solutions to a set of coupled first-order differential equations. Each dependent variable is plotted on one axis. The curves on a phase portrait, usually called "trajectories," show possible behaviors of the system.

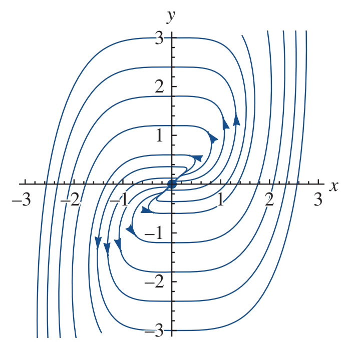
(We'll discuss below how to make phase portraits for second-order differential equations by writing them as sets of coupled first-order equations.)

Each point on a phase portrait represents a possible state of the system. Since the system could presumably start in any possible state, each point can also be said to represent a possible initial condition. For example, we can see from Figure 10.3 that if Romeo and Juliet start with any combination of feelings for each other that has a combined magnitude $R^2 + J^2 = 9$ (in whatever units we might use to measure feelings) then they will oscillate with that same magnitude forever.

If two trajectories crossed each other, one set of initial conditions could lead to two possible outcomes. That is technically possible for non-linear equations, but it takes work to contrive such an example. We will assume that phase portrait trajectories never cross.

EXAMPLE **A Phase Portrait**

Problem:
The figure below is a computer-drawn phase portrait for the equations $\dot{x} = x - y$, $\dot{y} = x^3$. Based on this drawing, describe the possible long-term behaviors of the system.



Solution:
The overall trend is a counterclockwise rotation. Whenever x is positive, y is increasing; in the upper left of the plot x is decreasing and in the lower right x is increasing. Take a glance at the differential equations and convince yourself that this is just what we would expect.

But we also see something else: no matter what initial conditions the system starts in, the amplitude of the oscillation increases over time. The system spirals out toward infinity.

Critical Points and Separatrices

We are going to analyze the behavior of the following system by drawing a phase portrait.

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - 1 \tag{10.3.1}$$

The first thing we always look for are the points where $x'(t) = y'(t) = 0$. If the system ever reaches such a "critical point" it will stay there forever. Physically these represent equilibrium states of the system.

Definition: Critical Point

A set of first-order differential equations for the functions $x_1(t)$, $x_2(t)$, … has a critical point at a set of values $(x_1, x_2, \ldots)$ if all of the derivatives $\dot{x}_1$, $\dot{x}_2$, … equal zero there.

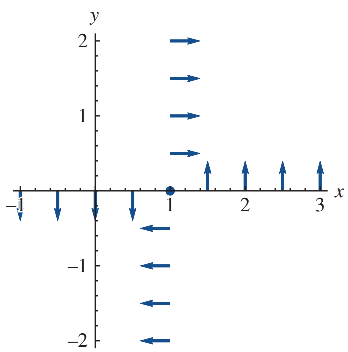
A critical point is "stable" (or "attractive") if all the trajectories near that point converge toward it. A critical point is "unstable" (or "repulsive") if all the trajectories near that point move away from it. It is also possible for a critical point to be neither attractive nor repulsive—either because some nearby trajectories approach it and others move away, or because nearby trajectories orbit around the critical point as in Figure 10.3.

Critical points are the only places where a trajectory can begin or end. A trajectory can come in from infinity and end at a critical point, start from a critical point and go off to infinity, start and end at infinity, start and end at critical points, or make a closed loop.

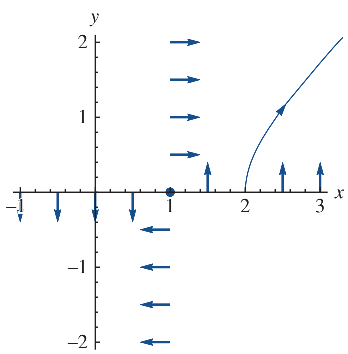
It isn't hard to determine that Equations 10.3.1 have a critical point at $(1, 0)$ and nowhere else. So we will start building our phase portrait out from there.

Along the x-axis we have $x' = 0$ and $y' = x - 1$. So to the right of our critical point the trajectories point straight up, and to the left they point straight down.

Along the line $x = 1$ we have $x' = y$ and $y' = 0$. So above our critical point the trajectories point directly to the right, and below they point left. Let's draw what we have so far.

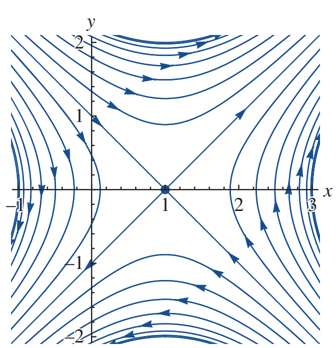


Based on just those arrows and a little bit of thought we can say a surprising amount about this system. For instance, suppose the initial conditions are $x = 2$, $y = 0$ (directly on one of our arrows, just to make the first step easy). The initial movement will be straight up on the graph: that is, y will increase $(y' = 1)$ while x holds steady $(x' = 0)$. So a short time later finds us at the point $(2, \Delta y)$. Now y' is still 1, but x' is now a small positive number. Our graph starts to veer slightly to the right. As we move higher the increasing y causes x' to increase, and the increasing x causes y' to increase. So our graph will head toward (∞, ∞). In Problem 10.37 you will sketch in a few other curves using a similar process. You may be able to predict much of the behavior just by looking at the drawings we've already done. Trajectories in the upper-right-hand corner will generally head up and right, as our example above did. Trajectories in the lower-left-hand corner will generally head down and left.



Where is the dividing line between these two destinations? In Problem 10.38 you will show that the ultimate destiny of any trajectory in this system depends on what side of $y = 1 - x$ the initial conditions fall on. Any path above this line will eventually head toward (∞, ∞), while any path below it will head toward $(-\infty, -\infty)$. We say that $y = 1 - x$ is a "separatrix" for this phase portrait, because it separates two regions that exhibit qualitatively different behavior.

If you look closely at the phase portrait you can tell that $y = 1 - x$ is actually made up of two separatrices, each one a trajectory that comes in from infinitely far away and asymptotically approaches the critical point. Just like a critical point, a separatrix can be attractive, repulsive, or neither. The drawing shows that the separatrices along $y = 1 - x$ are "repulsors": trajectories near them move away from them over time. The drawing also shows another pair of separatrices along $y = x - 1$. They are "attractors": trajectories move toward them over time. All of the separatrices begin or end on the critical point $(1, 0)$, which is itself neither attractive nor repulsive.

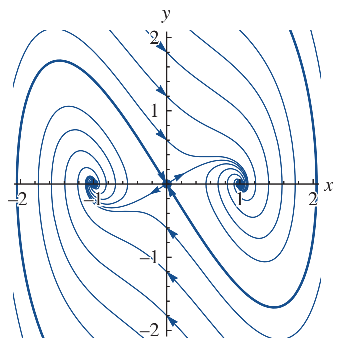


The phase portrait for $\dot{x} = y$, $\dot{y} = x - 1$

Drawing phase portraits by hand is not something we do a lot of. It's a worthwhile exercise to go through a few times because it gives you a greater appreciation for the really valuable skill, which is interpreting the behavior of a system from a given phase portrait.

EXAMPLE **More Critical Points**

Problem:
The drawing shows the phase portrait for $dx/dt = y$, $dy/dt = -x^3 + x - y$. Use this drawing to discuss the possible behaviors of the system.



Solution:
It's always helpful to start with the critical points. Setting the first equation equal to zero gives $y = 0$. The second one is zero when $x^3 = x$, with solutions $x = 0, \pm 1$. We can see the three critical points on the drawing.

Around those points the phase portrait shows two qualitatively different behaviors. The system can either spiral in toward the critical point at $x = 1$ or the one at $x = -1$. Dividing those two possibilities are a pair of separatrices, shown in bold. One approaches the origin from the upper left and the other from the lower right. (There's another pair of separatrices connecting the origin to the other critical points, but we're going to focus on the ones shown in bold.) It's far from obvious that the initial conditions $(1, 1)$ lead toward the point $(1, 0)$ and the initial conditions $(3, 1)$ lead toward the point $(-1, 0)$, but it's easy to see those things looking at the phase portrait.

On a technical note, it can often take a lot of trial and error with the computer to plot separatrices since you have to find just the right initial conditions. Later in this section we'll show you a trick that can help with that.

A lot of the point of the previous two examples is that critical points and separatrices reveal a tremendous amount about the possible behaviors of the system.

Second-Order Equations

We said above that phase portraits are for systems of coupled first-order differential equations. They can also be used for second-order equations, however, which include most of the equations used in physics. The trick is to write one second-order equation as two first-order equations. Suppose we have an equation of the form $x''(t) = F(x(t), t)$. We can always define a new function $v(t) = x'(t)$, and now we have the two coupled equations $x'(t) = v(t)$ and $v'(t) = F(x(t), t)$.

EXAMPLE **Damped Simple Harmonic Oscillator**

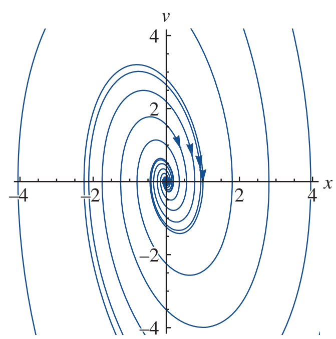
Problem:
A mass on a damped spring obeys the differential equation $\ddot{x} = -4x - \dot{x}$, where x and t are measured in SI units. Find all the critical points for this system, draw a phase portrait for it, and describe the possible behaviors.

Solution:
We begin by writing this as two first-order equations.

$$\dot{x} = v, \quad \dot{v} = -4x - v$$

The only critical point occurs at the origin: $x = v = 0$.

We had a computer graph trajectories for a set of initial conditions laid out along the unit circle in the first quadrant. We can see that all of the trajectories spiral in toward a stable equilibrium at the origin. This point represents $x = v = 0$: the mass is at rest at $x = 0$, as we would expect from a damped oscillator.



Remember that every point on a phase portrait represents one possible state of the system. For a pair of first-order equations for $x(t)$ and $y(t)$, a state is a value for each of those functions. For a second-order equation $\ddot{x} = \ldots$, a state is a value for x and a value for its first derivative—for instance, the position and velocity of the mass at any given moment—so those are the axes of the phase portrait.

Some Tips for Generating Phase Portraits on a Computer

In general, we can get a computer to make a phase portrait by giving it a list of initial conditions and having it numerically solve the differential equations for each one. Each solution will be a pair of functions $x(t)$, $y(t)$, and we can ask the computer to plot the parametrically defined curve (x, y) for each one. Finally we ask it to show all those curves together on one plot. Coming up with a good set of initial conditions can be tricky, and in the end there's no substitute for trial and error, but some guidelines can help.

We begin by finding all of the critical points and make sure that the initial conditions cover the region around those critical points. Sometimes a vertical or horizontal line of points can be useful, sometimes a circle of points around a critical point can help, and sometimes using a mix of the two can help.

Even with all that, the direction of the trajectories can be a problem. If all trajectories spiral in toward the origin and we start at a circle of points around the origin, our phase portrait won't show anything outside that initial circle. One way to deal with that is to solve the equations *backwards*. If our system is $\dot{x} = x + y$, $\dot{y} = x$ then we can generate a valid trajectory by starting at any initial point and solving $\dot{x} = -x - y$, $\dot{y} = -x$. That will trace the trajectory backwards from that point. If we plot the trajectory forwards and backwards from each point we will cover the phase portrait more effectively.

That backwards trick is especially useful for plotting separatrices that asymptotically approach critical points. If you look at the figure in "Example: More Critical Points" above, the separatrices are the only two trajectories that approach the critical point at the origin. Just choosing random initial conditions it could take forever to find one that just happens to be on one of the separatrices. So instead we chose two points near the critical point and evolved the system backwards from those points to plot the separatrices.

Stepping Back: Phase Space

Any physical system has some number of dependent variables. Most commonly these are functions of time. The state of the system at any given time consists of the value of each of those variables and, if they obey second-order differential equations, of their derivatives. Given a set of differential equations for those variables, we can predict the future behavior of the system from the initial conditions.

A phase portrait is a tool for visualizing those behaviors. Each axis is a dependent variable, and taken together those axes define all the possible states of the system. That space of all possible states is called "phase space." In principle, an object moving in 3 dimensions has a 6 dimensional phase space, because to specify its state we need to give all three components of its position and of its velocity. For simpler situations such as an object that can only move in 1 dimension, however, the phase space is 2 dimensional and a phase portrait can provide a useful visualization.

10.3.2 Problems: Phase Portraits

10.34 **Walk-Through: Phase Portraits.** Consider the system described by the equations $dx/dt = x + y$, $dy/dt = -2y$. You're going to draw a phase portrait for this system. By the time the problem is done you're going to have a lot on that drawing, so you might want to start by drawing a big set of axes going from -6 to 6 in both directions. When we ask where a trajectory begins or ends, remember that one possible answer is "at infinity."

(a) Find all critical points of this system. (That is, find all points where both $x'(t)$ and $y'(t)$ are zero.)

(b) Along the x-axis, $y'(t) = 0$.

- **i.** What does that imply about all trajectories that start along that axis? Draw arrows to represent the time evolution of the system along that axis. You will need to think about which way those arrows point.
- **ii.** The positive x-axis is a separatrix for this system. Is it an attractor or a repulsor? Explain how your answer comes from the equations. Where does this trajectory begin and end? (One of the answers will be "at infinity.")
- **iii.** The negative x-axis is another separatrix. Is it an attractor or repulsor? Where does it begin and end?

(c) Analyze the behavior in the first quadrant.

- **i.** At the point $(1, 1)$ the derivative in the x-direction is 2 and in the y-direction it's -2. If the system starts at that point, in what direction will it initially move in the xy-plane? Draw an arrow at $(1, 1)$ pointing in that direction.
- **ii.** What can you say about the direction of the trajectories *everywhere* in the first quadrant? Based on that, describe the behavior of any trajectory that starts in the first quadrant.
- **iii.** Sketch a trajectory beginning at the point $(1, 2)$. You don't need to be exact, but try to have the trajectory at each point going in roughly the correct direction, and be sure to show the right behavior in the limit $t \to \infty$.

(d) The second quadrant is more complicated.

i. If a trajectory starts slightly to the left of the positive y-axis, what direction will it move in initially? Where will that trajectory go at late times? Sketch one such trajectory, starting at $(-.1, 4)$.

ii. If a trajectory starts at the point $(-2, 2)$, what direction will it move in initially? As soon as it moves away from that point, what will happen to its direction? Where will that trajectory go at late times? Sketch the trajectory starting at $(-2, 2)$.

iii. Explain how your last two answers imply that there must be a separatrix in the second quadrant, passing somewhere in between the points $(-2, 2)$ and $(-.1, 4)$. Is this separatrix attractive or repulsive? Where does it begin and end?

iv. You don't have enough information yet to know the exact shape of this separatrix; you'll work that out in Problem 10.35. For now just sketch in a curve that matches the answers you've given about it. Include arrows showing which direction the separatrix trajectory goes.

v. You drew two trajectories that started in the second quadrant. Now extend them backwards, showing where they would have come from in order to reach the points $(-2, 2)$ and $(-.1, 4)$. As with all of these sketches your goal is to show the correct qualitative behavior, not to plot exact curves.

(e) Describe the behaviors of trajectories in the third and fourth quadrants. For each one you'll have to figure out if you can do it with a simple argument like the first quadrant or if it needs more careful work like the second one. When you're done you should have drawn in one more separatrix and a couple of other sample trajectories showing the possible behaviors of the system.

(f) If there are any regions of your phase portrait where it is not yet clear how the trajectories behave, sketch in enough trajectories to make it clear.

(g) Is the one critical point of the system attractive, repulsive, or neither?

(h) The separatrices divide the graph into four regions. For each region, indicate how trajectories starting in that region will evolve over time. What will the system approach as $t \to \infty$ in each case?

10.35 *[This problem depends on Problem 10.34.]* In Problem 10.34 you found roughly where the separatrices were by looking at the behavior of the system. In some cases that's the best you can do, but in this case you can find the separatrices analytically by guessing (correctly as it turns out) that they are lines. The key is that, for this particular system, the two separatrices you sketched are the only two trajectories that end on the critical point at the origin.

(a) If the system $dx/dt = x + y$, $dy/dt = -2y$ starts out at a point (x, y), what is the initial slope dy/dx of its trajectory? Your answer should be a function of x and y.

(b) Suppose the system starts at a point on the line $y = mx$. In order for the trajectory to stay on that line its initial slope would have to equal m. Using your answer to Part (a), write an equation expressing the statement "Starting at the point (x, mx), the trajectory's initial slope equals m."

(c) Solve that equation for m.

(d) What is the equation for the separatrices other than the positive and negative x-axes? (The separatrices are two halves of the same line, so they have the same equation.)

10.36 *[This problem depends on Problem 10.34.]* Have a computer make a phase portrait for the system $dx/dt = x + y$, $dy/dt = -2y$. Clearly indicate critical points and separatrices. Make sure your phase portrait has enough trajectories to see the behavior in each region, and that it includes arrows showing the directions of the trajectories.

10.37 Consider the system described by Equations 10.3.1 with initial conditions $x = 1$, $y = 2$.

(a) Calculate dx/dt and dy/dt at that point. Based on your answers, in what direction along the phase portrait will this trajectory initially move?

(b) After it moves in that direction for a short time, how will dx/dt and dy/dt change?

(c) By following the curve along in this way, trace the trajectory from that point.

(d) The equations $\dot{x} = -y$, $\dot{y} = -x + 1$ represent the same system evolving backward in time. Starting again at $(1, 2)$

trace in that direction to complete the trajectory.

(e) Draw a similar trajectory starting at the origin.

10.38 The Explanation (Section 10.3.1) claimed that Equations 10.3.1 have a repulsive separatrix along the line $y = 1 - x$. In this problem you will show how the equations lead to the behavior we saw in the computer-generated phase portrait.

(a) Show mathematically that at any point along this line $\dot{y} = -\dot{x}$. What does that imply about the time evolution of the system if the initial condition lies on that line?

Now consider three points. The point $B = (x_0, y_0)$ lies directly on the line $y = 1 - x$. The point $A = (x_0, y_0 + \Delta y)$ lies directly above that point, and $C = (x_0, y_0 - \Delta y)$ directly below.

(b) How does dx/dt at point A compare to dx/dt at point B?

(c) How does dy/dt at point A compare to dy/dt at point B?

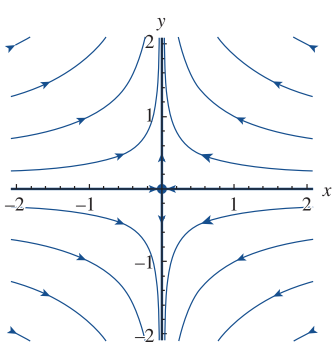
(d) We have seen what will happen to the system if it starts at point B. Based on that and your answers, what will happen to the system if it starts at point A? Will it move closer or farther from the line $y = 1 - x$?

(e) Repeat this analysis for point C.

10.39 *[This problem depends on Problem 10.38.]* Show that Equations 10.3.1 have an attractive separatrix along the line $y = x - 1$.

In Problems 10.40–10.42 you will be given a phase portrait. Classify all of the critical points and separatrices as attractive, repulsive, or neither. Describe how the system will evolve in time. (Your answer will almost always be of the form "if it starts in this region, then…")

10.40



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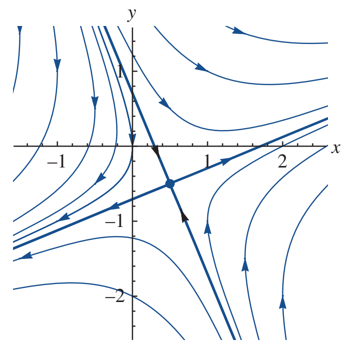

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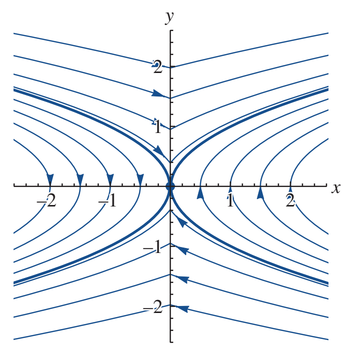


In Problems 10.43–10.50 sketch a phase portrait for the given set of equations. Your phase portrait should show all the critical points and enough trajectories to indicate the possible behaviors of the system.

10.43 $dx/dt = 4y,\ dy/dt = x$

10.44 $dx/dt = y,\ dy/dt = -x - y$

10.45 $dx/dt = y,\ dy/dt = -x + y$

10.46 $dx/dt = y,\ dy/dt = x - y$

10.47 $dx/dt = x + y,\ dy/dt = x - y$

10.48 $dx/dt = y,\ dy/dt = x^2 + y$

10.49 $dx/dt = y^2,\ dy/dt = -x$

10.50 $dx/dt = \rho,\ dy/dt = \phi$.

10.51 Given a pair of equations $x'(t) = f(x, y)$ and $y'(t) = g(x, y)$, write the equations that draw the same trajectories but in the opposite direction.

10.52 Given a pair of equations $x'(t) = f(x, y)$ and $y'(t) = g(x, y)$, write an equation for the slope dy/dx of the trajectory at the point (x, y).

In Problems 10.53–10.58 you will be given a second-order differential equation. Rewrite it as two coupled first-order equations. Have a computer draw the phase portrait for those two equations, and use that phase portrait to predict the possible time evolution of the system. A good answer would look like "If it starts with $x > 5$ moving to the right it will move in a positive direction forever. If it starts at $x > 5$ at rest or moving left slowly enough it will start moving right and continue that way forever. If it starts at $x < 5$ then…"

10.53 $x''(t) + 9x(t) = 0$

10.54 $x''(t) + 5x'(t) + 6x(t) = 0$

10.55 $x''(t) + 5x'(t) + 6x(t) = 2$

10.56 $x''(t) - 5x'(t) + 6x(t) = 2$

10.57 $x''(t) + x^3 = 0$

10.58 $x''(t) + \tan(4\pi x) = 0$

10.59 If you have a second-order differential equation for $x(t)$ you can draw a phase portrait for it where the axes are x and $\dot{x}$. Without knowing anything else about the system, what can you conclude about what the trajectories look like? In other words, what must be true about all phase portraits for second-order equations that is not necessarily true about phase portraits for coupled first-order equations?

10.60 Explain why a trajectory cannot begin or end at any point other than a critical point.

10.61 **Inflationary Cosmology** According to the theory of "inflation," the early universe went through a period during which virtually all of the energy was in the form of a "scalar field." You don't need to know what a scalar field is to solve this problem. All you need to know is that in the simplest model of inflation the field ϕ obeys the differential equation $\ddot{\phi} + m^2\phi + \sqrt{12\pi G(\dot{\phi}^2 + m^2\phi^2)} = 0$.

(a) Define $v = \dot{\phi}$ and rewrite this second-order equation as two coupled first-order equations for ϕ and v.

(b) What is the one critical point for this system?

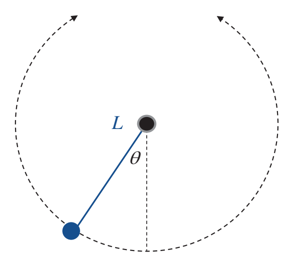
(c) Have a computer draw a phase portrait for the system. You can set the constants G and m equal to 1. There are two separatrices. Where do they begin and end? (In each case one of the answers is "at infinity.") Are they attractive or repulsive?

(d) For a typical trajectory, describe the evolution of the system. What happens at early times, middle times, and late times?

10.62 **Exploration: The pendulum.** A rigid undamped pendulum of length l obeys the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where θ is the pendulum's angle off the vertical and g is the gravitational constant. The usual approach is to say that $\sin\theta \approx \theta$ for small θ, which reduces the equation to the simple harmonic oscillator equation. This approximation only works for small θ, but the equation is difficult to solve more generally. (Go ahead and try. We dare you.)



(a) Rewrite this equation as two coupled first-order equations. What variables should go on the axes of the phase portrait for this system? Draw a set of axes with those labels. You'll fill in the phase portrait as you go through the problem.

(b) What are the critical points for the system? What physical states do these points represent? *Hint: Mathematically there are infinitely many critical points, but there are only two states they can represent.* Draw a few of these points on your plot.

(c) If the pendulum starts at the bottom ($\theta = 0$) and you give it a little push it will swing back and forth. Draw a set of trajectories on your plot representing this motion. Include arrows showing the direction of the trajectories.

(d) If instead you give the pendulum a large push it will swing in circles. Draw a set of trajectories on your plot showing motion in clockwise circles, and another set showing counterclockwise circles. (We said the pendulum is rigid, a rod instead of a string, so the bob always stays a distance L from the center, even when $\theta > \pi/2$.) Again, include arrows on the trajectories.

(e) You've drawn trajectories showing two types of motion, oscillations and circles. Draw separatrices on your plot in between those types of motion. Where do those separatrices begin and end? What do they represent physically?