10.13 Additional Problems

Throughout these problems, $H(x)$ refers to the Heaviside function and $\delta(x)$ to the Dirac delta function.

In Problems 10.294–10.315 solve the given differential equation.

- If no initial conditions are given, find the general solution. If initial conditions are given, find the specific solution corresponding to those initial conditions.
- Express your answer as a function $f(t)$ if it's easy to do so, but some answers are best expressed as an equation relating f and t.
- If your answer is a Laplace transform $F(s)$, leave it in that form.
- No computers should be required.

Hint: Some of the first-order equations are written with df/dt and some with dt and df separate. In some cases you may find it helpful to rewrite the equation the other way.

10.294 $d^2f/dt^2 + 5df/dt + 6f = 12$

10.295 $df/dt + 4t^3 f = t^3$, $f(0) = 9/4$

10.296 $df/dt = \sec(f + t^2) - 2t$

10.297 $d^2f/dt^2 + 25f = e^{5t}$, $f(0) = f'(0) = 1$

10.298 $d^3f/dt^3 + 4d^2f/dt^2 = 8$, $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$

10.299 $(f^3 + t^2)dt + (3f^2 t + 2f)df = 0$

10.300 $df/dt + t^2 f = t^2 f^4$

10.301 $(\sin t + f \tan t)dt - df = 0$

10.302 $(e^{f-t} + 1)dt - df = 0$, $f(0) = 0$

10.303 $f\left(\dfrac{df}{dt}\right) = \dfrac{t}{e^{f^2+t} + 2(f^2 + t)} - \dfrac{1}{2}$

10.304 $df/dt = (f^2 + t)/(f - 2tf)$, $f(0) = 1$

10.305 $(f/t - t^2 e^{-t})dt + (1/t + (f^2/t)e^{-t})df = 0$

10.306 $d^2f/dt^2 - df/dt + e^{2t} f = 0$

10.307 $d^2f/dt^2 + 4df/dt + 3f = \delta(t-1)$, $f(0) = 0$, $f'(0) = 2$

10.308 $df/dt = te^{f^2/t^2}/f + f/t$

10.309 $td^2f/dt^2 + df/dt - (4/t)f = 8/t$

10.310 $(t^2 f^2 + f)dt + t\,df = 0$, $f(1) = 1$. *Notice that the initial condition occurs at $t = 1$.*

10.311 $(f^2 + 2tf)dt - (tf + t^2)df = 0$

10.312 $df/dt + f/t = 2t^3 + 7$

10.313 $d^2f/dt^2 + f = \cos^2 t$

10.314 $d^2f/dt^2 + 2df/dt + f = \sin t$

10.315 $df/dt + 3f + 2\int_0^t f\,dt = e^{-t}$, $f(0) = 1$

In Problems 10.316–10.322, solve the given differential equation by substitution. Start by figuring out if the equation fits into one of the three special cases: Bernoulli, homogeneous, or $y(x)$ only appears inside derivatives. If it does, you immediately know the right substitution to use. If it doesn't, then you'll have to look at the equation and try to find the right substitution. (If you try something that doesn't work, try something else!)

10.316 $dy/dx = (y/x)^2 + (y/x)$

10.317 $e^y\left[(d^2y/dx^2) + (dy/dx)^2\right] + (1 - e^y) = 0$

10.318 $dy/dx = (y/x)^2 - (y/x)$. (Your substitution will lead you to a separable equation that can be integrated using partial fractions.)

10.319 $y''(x) = (y'(x)/x)^2 - (y'(x)/x)$. (Your substitution will lead you to a separable equation that can be integrated using partial fractions.)

10.320 $(d^2y/dx^2) - (dy/dx) - e^{2x}y = 0$

10.321 $xy(dy/dx) = x^2 + y^2$

10.322 $y(dy/dx) = x^2 + y^2$

10.323 (In this problem you will solve the differential equations you set up in the motivating exercise, Section 10.1 , but you do not need to have done that exercise to do this problem.) An object with mass $m = 1$ kg is attached to an ideal spring with spring constant $k = 9$ N/m, subject to a drag force $F_{drag} = -bv$ with $b = 6$ N·s/m, and also subject to an external driving force $F_e(t)$. In each case below the solution you write should be the general solution to the ODE.

(a) Write the differential equation for $x(t)$ assuming no driving force, $F_e = 0$. Solve the equation using guess and check.

(b) Write the differential equation for $x(t)$ assuming a driving force F_e that is a constant 3 N from $t = 0$ to $t = 10$ s. Solve it using Laplace transforms. Your final answer will be in the form of a Laplace transform $X(s)$, and will depend on the unspecified initial conditions x_0 and v_0.

(c) Write and solve the differential equation for $x(t)$ assuming an unspecified driving force $F_e(t)$. Your final answer will have the function $F_e(t)$ in it.

(d) Solve the differential equation for $x(t)$ assuming $F_e = t^5 e^{-3t}$.

10.324 In this problem you'll solve the equation $t^2\ddot{x} + t\dot{x} + x = 0$ twice.

(a) Find the general solution by using guess and check.

(b) Go back to the differential equation and find the general solution again, this time by using the substitution $u = \ln t$.

(c) You probably found two very different looking answers. In fact your guess-and-check answer should have been complex. Using properties of logs and exponents write that solution as a real function and show that it matches the solution you got from variable substitution.

10.325 The equation $x''(t) + 5x'(t) + 4x = 2\sin(2t)$ with initial conditions $x(0) = x'(0) = 0$ represents a damped, driven oscillator.

(a) Find the solution by using guess and check.

(b) Find the solution again by using variation of parameters. *Hint*: you may find the integrals on Page 485 useful.

(c) Find the general solution a third time by using a Laplace transform. Use a computer only to find the inverse transform at the end (and, if you want, to take an integral in the middle).

10.326 For each function below, draw the function and find its Laplace transform (by hand).

(a) $H(t-1) - H(t-2)$

(b) $H(t-4) - H(t-5)$

(c) $H(t-1) - H(t-2) + H(t-4) - H(t-5)$

(d) $H(t-1) + [H(t-2) - H(t-3)]e^{-t}$

10.327 Let $f(t) = (1/a)[H(t-2) - H(t-2-a)]$ where a is a constant.

(a) Draw a quick sketch of $f(t)$, assuming a is a reasonably small positive number.

(b) Find $\mathcal{L}[f(t)]$. Your answer will of course be in the form $F(s)$ but it will still have the constant a in it.

(c) Find $\lim_{a\to 0} F(s)$.

(d) What is $\lim_{a\to 0} f(t)$? Does your Laplace transform make sense for this function?

10.328 Let $f(t) = \sum_{n=0}^{\infty} H(t-n)$.

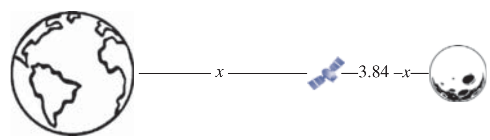
(a) Sketch $f(t)$.

(b) Find the Laplace transform $F(s)$. Your answer should be in closed form, i.e. with no summation.

10.329 The moon is about 384,000 km from Earth. A body in between the Earth and the moon experiences gravitational pulls in opposite directions from the two bodies. If we place the origin at the center of the Earth then the object's position obeys the differential equation

$$\frac{d^2x}{dt^2} = -\frac{2.98}{x^2} + \frac{.0366}{(3.84-x)^2}$$

measuring distance in 100,000s of kilometers and time in days. (You can do the problem in SI units, but the numbers are messier.)



(a) Explain why this equation is only valid for objects in between the Earth and the moon. In other words, how can you tell that it must give incorrect values if the object is at $x < 0$ (behind the Earth) or at $x > 3.84$ (beyond the moon)?

(b) Find the value of x at which the Earth's and moon's pulls exactly balance. Explain physically why you would expect this equilibrium value to be stable or unstable.

(c) Draw a phase portrait for this equation and confirm that there is an equilibrium point where you predicted and that it has the character you predicted.

10.330 **A Brief Voltage.** The charge buildup q on the capacitor in an RLC circuit obeys the following equation.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

Consider a circuit with $L = 1$ H, $R = 8\ \Omega$, $C = 1/15$ F, and $V(t) = 9\left[H(t-1/3) - H(t-2/3)\right]\sin(3\pi t)$ Volts.[16] The nature of this problem suggests

[16]These aren't particularly realistic inductance and capacitance values but it's harder to focus on the process when you're juggling numbers like 6×10^{-9} F.

Laplace transforms, but in this problem you will solve for $q(t)$ a different way. Assume $q(0) = 1$ C and $\dot{q}(0) = 0$.

(a) Sketch the function $V(t)$ for $t \geq 0$.

(b) For all $t < 1/3$ the voltage is 0. Use the method of guess and check to find the solution during this time period, subject to the initial conditions.

(c) Use your answer from Part (b) to find q and $\dot{q}$ at the moment when the voltage source turns on. If you punch your answer into a calculator, hold onto at least three digits of the answer.

(d) For the next 1/3 seconds the voltage is $\sin(3\pi t)$. Use the method of guess and check to find the solution during this time period. The initial conditions $q(1/3)$ and $\dot{q}(1/3)$ will come from your answer to Part (c).

(e) Use your answer to Part (d) to find q and $\dot{q}$ at the moment when the voltage source turns off.

(f) Find the solution after the voltage source turns off.

10.331 **Orthogonal Trajectories.** It is sometimes useful to find families of curves perpendicular to each other. For example, in two dimensions a set of charges creates field lines and equipotential curves that are orthogonal (perpendicular). If you know the formula for one set of curves, it's generally possible to find a differential equation that you (hopefully) can solve to find the orthogonal curves. As a first, simple example, consider the field created by a single point charge in 2D. You can measure that the equipotential curves are concentric circles: $x^2 + y^2 = r^2$. Each value of r corresponds to a different equipotential curve.

(a) What curves would you think would be perpendicular to those circles? Answer visually before going through the math.

(b) Use implicit differentiation to find the slope $m_{eq} = dy/dx$ of the equipotentials (circles) as a function of x and y.

(c) Since the field lines are orthogonal to the equipotentials, $m_{fl} = -1/m_{eq}$. Write that statement as a differential equation: dy/dx as a function of x and y for the field lines.

(d) Solve that differential equation to find $y(x)$ for the field lines. Your answer should have one arbitrary constant, so it represents a family of curves. Describe that family of curves. Does it match what you predicted?

(e) Now suppose a less symmetric configuration of charges led to elliptical equipotentials: $(x/2)^2 + y^2 = r^2$. Use the procedure outlined above to find $y(x)$ for the field lines. (Despite the name, "field lines" are not in general linear.)

10.332 *[This problem depends on Problem 10.331.]* Plot a representative sample of the elliptical equipotentials and their corresponding field lines from Part (e) of Problem 10.331.

10.333 *[This problem depends on Problem 10.331.]* Problem 10.331 walked you through the basic process of finding trajectories orthogonal to a family of curves, but it sidestepped a difficulty that can often arise. Consider as an example the curves $y = kx^4$, where k is the parameter that varies from one curve to the next (just as r was in Problem 10.331).

(a) First try finding the orthogonal trajectories exactly as you did in Problem 10.331. What you should find is an equation for $y(x)$ that includes *two* arbitrary constants, the original one k plus the new one that gets introduced by the integration. That doesn't make sense, so we need to make a slight change.

(b) Here's the problem. On each of our original curves k is a constant, and we treated it as such. But each orthogonal trajectory intersects many of the original curves, passing through many values of k, so on these new curves k is not a constant. We therefore have to eliminate k from this process before writing m_{fl}. So solve $y = kx^4$ for k and *then* take the derivative of both sides.

(c) Now finish the process. Find dy/dx for the new trajectories, separate variables, and integrate to find the orthogonal trajectories.