1.8 Differential Equations on a Computer

Some students consider computer use “cheating” in some mathematical sense, and believe that Real Scientists find all their solutions by hand. Other students rely too heavily on computers, believing they don’t need to learn how to solve equations on their own.

Practical scientists use computers as a tool to solve problems efficiently. They know when and how to use them, and what to do with the answers they get. They also know when to rely on their own understanding of the math.

This section will introduce some of the ways computers can handle differential equations. It will *not* discuss specific commands or syntax, because we don’t know what computer program(s) you happen to be using. We’re introducing the concepts here, but you will need to learn how to use a particular program, either from your class or from other resources.

1.8.1 Discovery Exercise: Differential Equations on a Computer

1. Go to `http://www.wolframalpha.com`.
2. Type the following:

```
solve dy/dx = e^x / sqrt(ky)
```

3. Note that the program shows you your question, written in standard mathematical notation, so you can make sure it interpreted your question correctly. The program also gives you solutions. Test one solution to confirm that it solves the differential equation.
4. Type the following:

```
solve dy/dx = y/(1-x^3)
```

5. Once again, the program gives you a solution. Perhaps you’d better take its word for this one.
6. Choose three other differential equations, ranging from easy ones that you know how to solve to more complicated ones that you don’t. Record your questions and the solutions. If you try one that the computer can’t solve just try a different one instead. If you get an answer that includes a function you’ve never heard of, that’s fine. We’ll discuss in the section below what to do with answers like that.

1.8.2 Explanation: Differential Equations on a Computer

If you spend the rest of your life studying methods of solving ordinary differential equations, it’s unlikely you will get better at it than computers are now. You ask the computer to solve $dy/dx + y = e^x y^2$ and it takes less than a second to identify “Riccati’s equation” and offer the general solution $y = e^{-x}/(C - x)$. (You can check it; it works.) After you’ve had this experience a couple of times, you will never want to go back to a world in which people looked up differential equations in huge tables.

On the other hand, don’t worry that computers are going to take your engineering job. Human beings are still needed to turn real-world problems into differential equations in the first place, and human beings are needed to interpret the results of those equations. The middle step, handling the differential equation, is where computers excel.

One way computers can help is by finding an “analytical solution”: a function with the requisite number of arbitrary constants that makes the differential equation true. In the above example the computer solved Riccati’s equation; you can now easily plug in an initial condition to find a specific function.

Unfortunately, some solutions can only be expressed as integrals or as infinite series, and may be quite difficult to work with. Worse still, you often can’t find any analytical solution, even with a computer! That doesn’t mean there’s anything wrong with the differential equation. It presumably still has a solution for any given set of initial conditions;[6] you simply can't write that solution in terms of known functions. In such cases, you can often find practical answers by asking the computer for a "numerical solution."

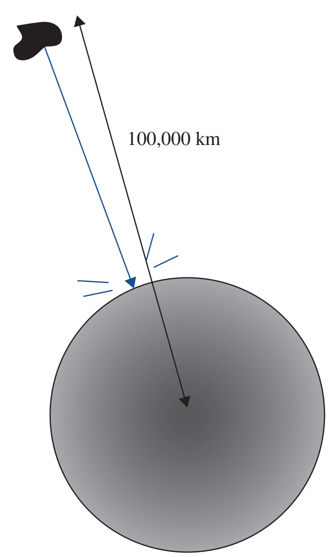
You are already familiar with one numerical technique, which is solving an integral by doing a Riemann sum. (This is how many calculators find integrals, which is why they know that $\int_1^2 2x\,dx = 3$ but may not know that $\int 2x\,dx = x^2 + C$.) A Riemann sum allows you to evaluate any definite integral to any desired degree of accuracy, *without* clever tricks or symbolic manipulation, but *with* dozens (or thousands) of computations, so it is ideally suited to a computer.

What does it mean to approach a differential equation numerically? Consider that a first-order differential equation tells you how much a function is changing at any given point. So once you have an initial condition, the differential equation tells you how the function will move up or down from that (known) point until you reach another (unknown) point. When you ask a computer to numerically solve a differential equation it generally uses some variation of this method, repeated many times over tiny intervals. We illustrate the use of numerical solutions below, and you will learn one such technique by hand in Problem 1.177.

Analytically or numerically, computers can be tremendously helpful in many different situations, once you learn to use them properly. The three examples below illustrate some of the situations you may run into, and some of the ways you may choose to handle them.

EXAMPLE **Mass Drivers**

Problem:
Centauri Prime won its war with rival planet Narn by hurling rocks from space. The rocks fell toward Narn under the influence of the planet's gravity, obeying Newton's law of universal gravitation $F = Gm_1 m_2/r^2$. Assume the mass and radius of Narn are comparable to those of the Earth, but that Narn's atmosphere exerts negligible drag. If a rock starts at rest 100,000 km from the center of Narn, how long will it take to strike the planet's surface?



[6]This is not guaranteed mathematically, but for real systems it can often be justified on physical grounds.

Solution:
The first step, of course, is setting up the equation. Using $F = ma$ and plugging in appropriate constants leads you to the equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2} \tag{1.8.1}$$

where $k = 4 \times 10^{14}\text{m}^3/\text{s}^2$. So now you plug that equation into a computer, sit back, and wait for it to solve all your problems! The bad news comes almost immediately: the computer can't find a function that happens to satisfy that particular differential equation. (This usually means there isn't one.) The analytical approach fails in this case.

This problem is ideal for a numerical solution, however. You need to look up the syntax for solving differential equations numerically on your computer system. Whatever system you are using you will need to tell the computer that:

- $d^2x/dt^2 = -4 \times 10^{14}/x^2$, and
- $x(0) = 10^8$, and
- $x'(0) = 0$

The computer generates a list of numbers representing x at different times t. You can make a plot of this function to see where it crosses the line $x = 6 \times 10^6$ (the radius of the planet) or you can use other computer functions to find that value for you. The answer you get is roughly $t = 55,000$. Since you put everything in SI units this value is in seconds, so as a last step you can divide it by 3600 to get the somewhat more useful answer $t \approx 15$ hours.

EXAMPLE An analytical solution that doesn't help

Problem:
A physical quantity is described by the equation:

$$\frac{ds}{dt} = s^3 - 2s^2 - s + 2$$

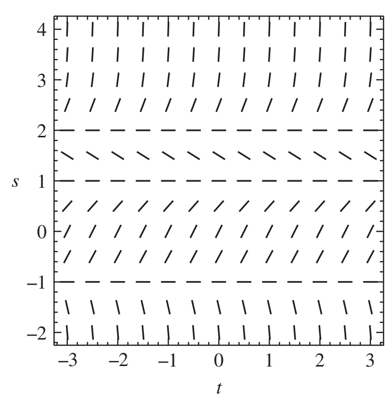
What are the possible long-term behaviors of the system?

Solution:
Unlike the previous example, this can be approached analytically, provided you happen to notice that the expression on the right can be factored. You separate variables, rewrite the left side using partial fractions, integrate, use the laws of logs, and simplify cleverly, and you arrive at $(s+1)(s-2)^2/(s-1)^3 = Ce^{6t}$. Now it's time to solve that for s … and at this point, let's say you start over and throw the problem at a

computer. It comes back almost immediately with an analytical answer:

$$-\frac{\dfrac{\sqrt[3]{\sqrt{e^{6(C+t)}\left(e^{6(C+t)}-1\right)^3}-2e^{6(C+t)}+e^{12(C+t)}+1}}{e^{6(C+t)}-1}}{\dfrac{1}{\sqrt[3]{\sqrt{e^{6(C+t)}\left(e^{6(C+t)}-1\right)^3}-2e^{6(C+t)}+e^{12(C+t)}+1}}}+1$$

That solution, seeded with different values of C, represents every possible function that this quantity could follow, and therefore contains all the information about the long-term behaviors of this quantity.[7] It's easy to look at that solution and in mere minutes conclude that you no longer care about the long-term behaviors of this quantity and might prefer to go into dentistry. A better solution (not to imply anything wrong with dentistry) is to ask the computer to draw a slope field.



The solution is now clear. If s starts between −1 and 2 it will asymptotically approach $s = 1$. If it starts above 2 or below −1 it will rise or fall (respectively) without bound. The computer can draw the slope field that tells you this, but *you* have to figure out that a slope field will be more useful than an analytic solution for this situation. You might wonder how we knew that drawing the slope field in the range $-2 \le y \le 4$ would show us all the possible behaviors. You can try to figure this out with trial and error, but a more surefire method is to start by finding the equilibria. Recall that these are points where $ds/dt = 0$, so you can have a computer tell you that −1, 1, and 2 are the roots of $s^3 - 2s^2 - s + 2$. Then you know that any range that includes those three points will be sufficient.

[7]Actually, it's a bit worse than that because this is one of three solutions to this equation. If you include all possible values of C in all three solutions then you have all the possible behaviors of $s(t)$. Remember that for non-linear equations a solution with enough arbitrary constants isn't guaranteed to be the general solution.

EXAMPLE

An analytical solution with unknown functions

Problem:
A function is known to obey the differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 9)y = 0 \ (x \geq 0)$$

where $y(0) = 0$. It is important to know for what *other* x-values this function equals zero. (This function represents an important class of equations that come up in mechanics, quantum mechanics, and probability. You'll explore some of its applications in the problems.)

Solution:
When you hand this differential equation to a computer, it comes back with something like this.

$$y(x) = C_1 J_3(x) + C_2 Y_3(x)$$

Huh?

We arrive now at one of the most important rules for using computer math systems: *when in doubt, look for help!* Look at the built-in help resources of your computer math program, or search online. It doesn't take long to find that $J_3(x)$ and $Y_3(x)$ are "Bessel functions." Don't be put off if you've never heard of one; look at a few graphs or properties and see if you can find the answers you need. In this case, $J_3(0) = 0$ and $Y_3(0)$ is undefined; since our problem stipulated that $y(0) = 0$, we must have $C_2 = 0$, leaving $y(x) = C_1 J_3(x)$.

Now, what about those zeros? Poking around a bit more, we find that J_3 has an infinite number of zeros. The first few are roughly 6.38, 9.76, and 13. Most computer math systems have a built-in function for generating as many of them as you need.

Analytical and Numerical Approaches

"Analytical" and "numerical" represent two very different approaches to problems involving differential equations. The computer can help with both approaches, but it *cannot* suggest which one you should use, so it's worth taking a moment to contrast them.

Disadvantages of analytical solutions include the following:

- Sometimes they are very messy.
- Sometimes they don't exist at all.
- Sometimes the answer you're looking for is just one simple number ("How long can the reactor run before all the fuel is exhausted?") and an analytical solution makes you work much harder to get it.

Disadvantages of numerical solutions include the following:

- To find a numerical solution you must specify initial and/or boundary conditions; there is no such thing as a *general* numerical solution to a differential equation. If you want to know the solution for different sets of initial conditions you have to find a separate numerical solution for each one.
- Similarly, all constants in your equations must be given numerical values. If you want to know how long it will take for a lunar lander to touch down on the surface of the moon you might find a numerical answer, but if you want to find an answer that applies

to landing on other moons or planets, or uses a different fuel burn rate or different size lander, you have to solve the equation again from scratch.

In cases where an analytical solution doesn't exist and you want to describe all the possible behaviors of your system, you have two main options. The first is to approach the problem graphically, using either slope fields (as described in this chapter) or phase portraits (described in Chapter 10). The second option is to approximate the differential equation with one that describes your system reasonably well, but which can be solved analytically. We discuss this method in Chapter 2.

Stepping Back

You can take entire courses or read volumes on computer approaches to differential equations, and this section is no substitute for all that time—or for the time you need to spend familiarizing yourself with the abilities and quirks of your particular software. Our main point here has been that a computer is an invaluable tool, but it is not a permission slip to turn off your brain; on the contrary, using the computer properly can require as much thinking and understanding as solving problems by hand.

But you should also be aware that knowing how to use the computer is quite different from knowing what it is doing "under the covers." Problem 1.177 walks you through an example of a numerical technique called Euler's method. Euler's method turns out to have a lot of drawbacks; modern software tends to use more complicated algorithms that are *based* on Euler's method but provide more accurate answers more quickly. But when you walk through even our brief introduction to that method, you get a beginning sense of what numerical solutions are about. If you find yourself having to program your own solutions (which is more common than you might suppose), you will need to learn the appropriate numerical recipes.[8] As obvious as this sounds, it's important to remember that the computer is not doing anything magic; it is doing the calculations that someone programmed it to do, and you can understand all those calculations even though you could never do so many so quickly.

1.8.3 Problems: Differential Equations on a Computer

Nearly all of the problems in this section require a computer. Whatever software you are using, remember that the first important skill is using the built-in help resources to identify the specific syntax required to solve these kinds of problems!

For any problem in this section where you find an answer numerically or by looking at a plot, you should give an answer that is accurate to at least two significant figures.

In Problems 1.159–1.163 you will be given a differential equation, a set of initial conditions, and a final time. Solve the equation numerically, plot the solution, and find the value of $y(t_f)$.

1.159 $dy/dt = t^3 - y^3$, $y(0) = 1/10$, $t_f = 2$

1.160 $d^2y/dt^2 = -y^3$, $y(0) = 1/10$, $y'(0) = 0$, $t_f = 2$

1.161 $d^2y/dt^2 = -\sin(y)$, $y(0) = 0$, $y'(0) = 1$, $t_f = 2\pi$

1.162 $d^2y/dt^2 + (dy/dt)^2 + y = 1$, $y(0) = 0$, $y'(0) = 0$, $t_f = 5$

1.163 $d^3y/dt^3 = -y^2$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = .2$, $t_f = 2$

1.164 For the differential equation $f''(x) = -(1/x)f(x)$ with boundary condition $f(0) = 0$ how many times does the solution cross the x-axis in the range $0 < x < 100$? Notice that $x = 0$ is not included in this range, so you shouldn't count the initial condition at the origin as one of the axis crossings. (*Hint*: Because this is a second-order equation with only one boundary condition the solution will have an arbitrary constant in it. You should still be able to answer this question.)

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[8] For a wonderful introduction to many types of computer algorithms for numerical calculations see *Numerical Recipes: The Art of Scientific Computing* by Press, et. al.

1.165 For the differential equation $y''(x) + xy'(x) + y(x) = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, the solution rises to a maximum, then falls, and then decreases toward zero. Find the value of x where this solution reaches its maximum value.

1.166 For the differential equation $y''(x) + y'(x) + xy(x) = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, find the first positive value of x at which the solution $y(x)$ equals zero.

1.167 Important Quantity I follows the differential equation $dI/dt = I^4 - 6I^3 + 9I^2 - 4I$.

(a) Have a computer find the solutions to the equation $I^4 - 6I^3 + 9I^2 - 4I = 0$. Explain how you know that those solutions are the equilibrium values of I.

(b) Have a computer generate a slope field with a range of values for I that includes all the equilibrium values you found in Part (a).

(c) Using this slope field, predict the long-term behavior of I. Your answer will consist of several different statements of the form "If I starts in this range, then it will head toward…"

1.168 Find all the equilibrium values of the differential equation $dx/dt = 4x^4 - 4x^3 - 4x^2 + 4x$ and classify each one as stable, unstable, or neither.

1.169 Consider the differential equation $y'(t) = \sin y$.

(a) Have a computer solve this equation analytically.

(b) Based on your solution, what is $\lim_{t\to\infty} y(t)$ if $y(0) = \pi/2$?

(c) What are the equilibrium values for this equation? *Hint*: There are an infinite number of them.

(d) Draw a slope field for this equation. You can do this by hand or with a computer. Your graph should show at least three equilibrium values.

(e) Make sure your slope field confirms your answer to Part (b) and then use it to find $\lim_{t\to\infty} y(t)$ if $y(0) = -\pi/2$.

1.170 Consider $y'(t) = \sqrt{y - y^2}$

(a) Have a computer solve this analytically. Verify that the solution works.

(b) Draw a slope field for this equation for $0 \le y \le 1$. (Why did we have to restrict it to that range?)

(c) You may have found that the analytic solution and slope field seem to predict very different behavior. Explain. *Hint*: when you verified the analytic solution, what assumptions did you have to make?

(d) Describe the long-term behavior of $y(t)$ if $y(0) = 0$. This is a trick question because there is more than one possible answer. Give at least two. This is an example of the general fact that non-linear equations don't always have a unique solution for each initial condition.

(e) Describe the long-term behavior of $y(t)$ if $y(0) = 1/2$. There is only one answer for this condition.

1.171 An asteroid is detected heading straight toward Earth at 25 km/s. When it is first detected it is 500,000 km from the center of the Earth.

(a) How long will it take to reach the surface of the Earth? (*Hint*: You can find all the information you need in the "mass drivers" example on Page 11. Be careful to convert all units to SI before entering equations on a computer.)

(b) How long would it take an asteroid to reach the surface of Jupiter if were moving straight toward it at 25 km/s starting 500,000 km from the center of Jupiter? (*Hint*: The constant k in Equation 1.8.1 is the universal constant G times the mass of the planet, so you can calculate its value for Jupiter by looking up G and the mass of Jupiter. You will also need to look up the radius of Jupiter to solve the problem.)

1.172 *[This problem does not require a computer.]* In the example on Page 14 we solved the equation

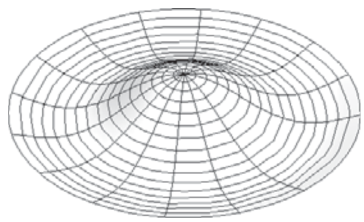
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 9)y = 0 \ (x \ge 0)$$

with the boundary condition $y(0) = 0$. Explain why we could have arrived at the same solution if all we had specified was that $y(0)$ must be finite. Using similar logic, explain why you *cannot* solve this equation with the boundary condition $y(0) = 1$.

1.173 A circular drumhead of radius R can have circular standing waves whose amplitude as a function of distance from the center $A(\rho)$ obeys the differential equation

$$A''(\rho) + \frac{1}{\rho}A'(\rho) + k^2A(\rho) = 0 \qquad (1.8.2)$$

where k is a constant related to the frequency of the wave. The boundary conditions for this equation are that the edges of the drum are clamped down, meaning $A(R) = 0$, and that $A(0)$ must be finite.



(a) Find the general solution to this differential equation. The result should be two special functions, each multiplied by an arbitrary constant.

(b) Using the condition that $A(0)$ must be finite explain why one of the two arbitrary constants in your solution must be zero. Write the resulting solution with one arbitrary constant.

(c) The condition that $A(R) = 0$ does *not* restrict your other arbitrary constant. Instead it restricts the possible values of k. By looking up the values at which the function you found equals zero, find the first three possible values of k for which the condition $A(R) = 0$ can be satisfied for a drum of radius $R = 0.1$ m.

(d) The frequency f is related to k by $f = kv/(2\pi)$, where v is the speed of sound on the drumhead (which depends on its tension). For a drumhead of radius 0.1 m with sound speed $v = 100$ m/s find the first three possible frequencies for circular waves. As a check on your work your answer should come out in units of 1/s, otherwise known as Hertz (Hz).

(Drums can also have waves with more complicated shapes. We'll consider the general solution for a vibrating drumhead in Chapter 11.)

1.174 *[This problem depends on Problem 1.173.]* A circular wave on a drumhead is described by the solution you found in Problem 1.173 multiplied by $\cos(2\pi ft)$, where f is the frequency you found at the end. Using the third value of k you found and the corresponding value of f, make a series of nine plots similar to the one at the beginning of Problem 1.173, each plot showing the drumhead at a different time. Your final plot should be at the time when it returns to its original shape. (If your program can make animations you can do a single animation instead of the sequence of nine plots.) Use 0.2 for your arbitrary constant (which gives the amplitude of the wave).

1.175 In quantum mechanics a particle is described by a "wavefunction" ψ that tells you the probabilities of finding the particle in different places. For a particle in a spherical region with no forces acting on it the wavefunction obeys the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} - \left(\frac{l(l+1)}{r^2} - 1\right)\psi(r) = 0$$

where l is an integer related to the particle's angular momentum and the distance from the origin r is expressed in units that allow you to eliminate all other constants from the problem.

(a) Find the general solution for $\psi(r)$.

(b) Using the condition that $\psi(0)$ must be finite, set one of the arbitrary constants in your general solution to zero and write the remaining solution.

(c) The values of r where $\psi(r) = 0$ indicate radii where there is zero chance of finding the particle. Find the first such non-zero radius for the three cases $l = 0$, $l = 1$, and $l = 2$.

1.176 Superman's enemy Lex Luthor is holding a block of kryptonite, which is deadly to Superman. Superman is attempting to reach Luthor, but the closer he gets to the kryptonite the slower he moves. Assume his velocity is given by $v = -v_s x/(d + x)$ where v_s and d are constants and x is Superman's distance from the kryptonite.

(a) Sketch the function $v(x)$ and describe what happens to Superman's speed when he is very far from the kryptonite and when he is very close.

(b) Try to solve this differential equation by hand using separation of variables to find the function $x(t)$. Explain why this *doesn't* work.

(c) Assume Superman's normal speed when he is far away from kryptonite is 1000 m/s (faster than a speeding bullet) and that his speed drops to half that value when he is 20 m from the kryptonite. Find the values of v_s and d and solve for $x(t)$ numerically assuming he starts 100 m away. If he needs to get within 1 meter of the kryptonite before he can reach it and get rid of it, how long will that take him? (Be careful with signs!)

1.177 Exploration: Euler's Method By Hand *[This problem does not require a computer.]*
The simplest technique for numerically solving differential equations is "Euler's method." In this problem, you will use Euler's method to answer the question: If $dy/dx = y$ and $y(0) = 1$, what is $y(1)$?

This problem asks us to follow the function from $x = 0$ to $x = 1$. We choose to take this journey in three steps, each of length 1/3.

(a) You're going to draw a slope field for this equation. To start, draw the axes and draw a small segment of slope at the point $(0, 1)$. Explain how we know that the value of the slope at that point must be 1.

(b) Our next task is to find the value $y(1/3)$. Using the slope you found, explain how we can know that $y(1/3) \approx 4/3$.

(c) Calculate the slope of the curve at $(1/3, 4/3)$ and use the result to draw the next entry in your slope field at that point.

(d) Using the slope you just found, estimate the value of $y(2/3)$.

(e) Calculate the slope at the new point you just found and use it to draw another entry on your slope field.

(f) Use the last slope you calculated to estimate $y(1)$.

(g) We began with the question "What is $y(1)$?" and you answered that question in Part (f). Explain why your answer is not exactly correct. Your explanation should enable you to predict whether the actual $y(1)$ is higher or lower than your approximation.

(h) Solve the equation $dy/dx = y$ with initial condition $y(0) = 1$ analytically and find the exact, correct value for $y(1)$. If you didn't correctly predict whether your approximate answer would be too high or too low rethink your explanation and explain the result you did get.

1.178 Exploration: Euler's Method By Computer *[This problem depends on Problem 1.177.]*

In Problem 1.177 you used Euler's method to find an approximate value for $y(1)$ given the equation $dy/dx = y$ and the initial condition $y(0) = 1$. You did this in three steps and found a not-so-great approximation to the exact answer.

(a) Have a computer repeat the calculation, but this time using 10 steps. In other words, starting from the known value $y(0) = 1$ calculate the slope $y'(0)$ and use that to find an approximate value for $y(0.1)$. Then use that to find the slope $y'(0.1)$ and thus the value $y(0.2)$, and so on until you have found a value for $y(1)$. Record the resulting value for $y(1)$. *Hint*: We strongly suggest using a loop to do the ten calculations rather than writing all ten of them out one at a time. This will be faster and easier in this step, and essential for the next one.

(b) Repeat Part (a) with 20 steps instead of 10. You should find that as you increase the number of steps your answer gets closer to the exact answer. *Hint*: If you haven't done so already you should be able to write your calculation in such a way that you can change the number of steps simply by changing one number and rerunning.

(c) Keep doubling the number of steps until your answer for $y(1)$ is within 1% of the exact answer. How many steps do you need and how close is the resulting answer to the exact one?

You've now used Euler's method to get a fairly accurate answer to a problem that you could have answered more easily without it anyway. Of course, the real power of the method is in solving problems you couldn't easily solve analytically! So now consider the equation $dy/dx = \tan(x + y)$ with initial condition $y(0) = 1$.

(d) Ask your computer to *analytically* solve this differential equation. What result do you get?

(e) Use Euler's method to find $y(0.1)$ with four steps. In other words use the slope $y'(0)$ to estimate $y(.025)$ and so on.

(f) Try again with eight steps and keep doubling the number of steps until you get an answer that differs from its predecessor by less than 1%.