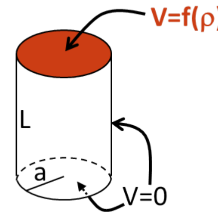
Discovery Exercise for Separation of Variables—Polar Coordinates and Bessel Functions

This exercise requires that you look up the ordinary differential equations for Bessel functions and "modified Bessel functions" and some of the basic properties of those functions.

A vertical cylinder of radius a and length L has no charge inside. The cylinder wall and bottom are grounded: that is, held at zero potential. The potential at the top is given by the function $f(\rho)$. Find the potential inside the cylinder.



The potential in such a situation will follow Laplace's equation $\nabla^2 V = 0$. The system geometry suggests that we write this equation in cylindrical coordinates. The lack of any ϕ-dependence in the boundary conditions lends the problem azimuthal symmetry: that is, $\partial V/\partial\phi = 0$. We can therefore write Laplace's equation as:

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1}$$

with the boundary conditions:

$$V(\rho,0) = 0, \quad V(a,z) = 0, \quad V(\rho,L) = f(\rho)$$

1. To solve this, assume a solution of the form $V(\rho,z) = R(\rho)Z(z)$. Plug this guess into Equation 1.

2. Separate the variables in the resulting equation so that all the ρ-dependence is on the left and the z-dependence on the right.

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3. Let the separation constant equal zero.

   (a) Verify that the function $R(\rho) = A\ln(\rho) + B$ is the general solution for the resulting equation for $R(\rho)$. (You can verify this by simply plugging our solution into the ODE, of course. Alternatively, you can find the solution for yourself by letting $S(\rho) = R'(\rho)$ and solving the resulting first-order separable ODE for $S(\rho)$, and then integrating to find $R(\rho)$.)

   (b) We now note that within the cylinder, the potential V must always be finite. (This type of restriction is sometimes called an "implicit" boundary condition.) What does this restriction say about our constants A and B?

   (c) Match your solution to the boundary condition $R(a) = 0$, and show that this reduces your solution to the trivial case $R(\rho) = 0$ everywhere. We conclude that, for any boundary condition at the top other than $f(\rho) = 0$, the separation constant is not zero.

4. Returning to your equation in Part 2, let the separation constant equal k^2.
   (a) Solve for $R(\rho)$ by matching either the ODE for Bessel functions or the ODE for modified Bessel functions. The constant p will determine the order of the resulting solutions. The result will be two solutions, each with an arbitrary constant in front.
   (b) Use the implicit boundary condition that $R(0)$ is finite to eliminate one of the two solutions.
   (c) Explain why the remaining solution cannot match the boundary condition $R(a) = 0$. You can do this by looking up the properties of the function in your solution or you can plot the remaining solution, choosing arbitrary positive values for k and the arbitrary constant, and explain from this plot why the solution cannot match $R(a) = 0$.

5. Returning to your equation in Part 2, let the separation constant equal $-k^2$.
   (a) Solve for $R(\rho)$ by matching either the ODE for Bessel functions or the ODE for modified Bessel functions. The constant p will determine the order of the resulting solutions. The result will be two solutions, each with an arbitrary constant in front.
   (b) Use the implicit boundary condition that $R(0)$ is finite to eliminate one of the two solutions.
   (c) Use the boundary condition $R(a) = 0$ to restrict the possible values of k.

6. Based on your results from Parts 3–5 you should have concluded that the separation constant must be negative to match the boundary conditions on $R(\rho)$. Calling the separation constant $-k^2$, solve the equation for $Z(z)$, using the boundary condition $Z(0) = 0$.

7. Write $V(\rho, z)$ as an infinite series.

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The process you have just gone through is the same as separation of variables in Cartesian coordinates, but this ODE led to Bessel functions instead of sines and cosines. Bessel functions, like sines and cosines, form a "complete basis"—you can use them in series to build almost any function by the right choice of coefficients.