# Discovery Exercise for Separation of Variables—The Basic Method

Consider a bar going from $x = 0$ to $x = L$. The "heat equation" governs the evolution of the temperature distribution:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x,t)$ is the temperature and $\alpha$ is a positive constant called "thermal diffusivity" that reflects how efficiently the bar conducts heat. Both ends are immersed in ice water, so $u(0,t) = u(L,t) = 0$, but the initial temperature may be nonzero at other interior points. You are going to solve for the temperature $u(x,t)$. Your strategy will be to guess a solution of the form $u(x,t) = X(x)T(t)$ where $X$ is a function of the variable $x$ (but not of $t$), and $T$ is a function of the variable $t$ (but not of $x$).

1. Plug the function $u(x,t) = X(x)T(t)$ into Equation 1. Note that if $u(x,t) = X(x)T(t)$ then $\partial u/\partial x = X'(x)T(t)$.

2. "Separate the variables" in your answer to Part 1. In other words, algebraically rearrange the equation so that the left side contains all the functions that depend on $t$, and the right side contains all the functions that depend on $x$. In this problem—in fact in many problems—you can separate the variables by dividing both sides of the equation by $X(x)T(t)$. You should also divide both sides of the equation by $\alpha$, which will bring $\alpha$ to the side with $t$; this step is not necessary but it will make the equations a bit simpler later on.

3. The next step relies on this key result: if the left side of the equation depends on $t$ (but not on $x$), and the right side depends on $x$ (but not on $t$), then both sides of the equation *must equal a constant.* Explain why this result must be true.

4. Based on the result from Part 3, you can now turn your *partial* differential equation from Part 1 into two *ordinary* differential equations: "the left side of the equation equals a constant" and "the right side of the equation equals *the same* constant." Write both equations. Call the constant $P$.

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The next step, finding real solutions to these two ordinary differential equations, depends on the sign of the constant $P$. We shall therefore handle the three cases separately. We'll start by solving the equation for $X(x)$. Remember that the constant $\alpha$ is necessarily positive. Avoid complex answers.

5. Assuming $P > 0$, we can replace $P$ with $k^2$ where $k$ is any real number. Solve the equation for $X(x)$ for this case. Your solution will have two arbitrary constants in it: call them $A$ and $B$.

6. Assuming $P = 0$, solve the equation for $X(x)$, once again using $A$ and $B$ for the arbitrary constants.

7. Assuming $P < 0$, we can replace $P$ with $-k^2$ where $k$ is any real number. Solve the equation for $X(x)$ for this case, once again using $A$ and $B$ for the arbitrary constants.

8. For two of the three solutions you just found the only way to match the boundary conditions is with the "trivial" solution $X(x) = 0$. Only one of the three allows for nontrivial solutions that match the boundary conditions. Based on that fact, what must the sign of $P$ be?

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9. The boundary condition $u(0, t) = 0$ implies that $X(0) = 0$. Plug this into your solution for $X(x)$ to find the value of one of the two arbitrary constants.

10. Find the values of $k$ that match the second boundary condition $u(L, t) = 0$. *Hint*: there are infinitely many such values. We will return to these values when we match initial conditions. For the rest of this exercise we will continue to just write $k$.

Having found $X(x)$, we now turn our attention to the ODE you wrote for $T(t)$ way back in Part 4.

11. Replace $P$ with $-k^2$ in the $T(t)$ differential equation, as you did with the $X(x)$ one. (Why? Because both differential equations were set equal to the *same constant* $P$.)

12. Having done this replacement, solve the equation for $T(t)$. Your solution will introduce a new arbitrary constant: call it $C$.

13. Write the solution $u(x, t) = X(x)T(t)$ based on your answers. This solution should depend on $k$.

14. Explain why, when you combine your $X(x)$ and $T(t)$ functions into one $u(x, t)$ function, you can combine the arbitrary constants from the two functions into one arbitrary constant.

The solution you just found is a normal mode of this system. The method of separation of variables involves finding these normal modes, as you just did above, and then writing the general solution to such an equation as a linear superposition of all the normal modes. Then you can use initial conditions to solve for the arbitrary constants.