Discovery Exercise for Separation of Variables—Inhomogeneous Boundary Conditions

1. Separation of variables is simplest when a problem has only one inhomogeneous boundary or initial condition. Explain why the technique of separation of variables, in its simplest form, relies on this limitation.

Now consider a problem with three inhomogeneous boundary conditions. The cylinder below has no charge inside. The potential therefore obeys Laplace’s equation, which in cylindrical coordinates is:

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The potential on top is given by $V_T$, which could be a constant or a function of $\rho$ or even a function of $\rho$ and $\phi$. (For our present purpose it doesn’t really matter.) The potential on the bottom is given by $V_B$. The potential on the side is $V_S$ which could be a function of both $z$ and $\phi$.

2. The approach to such a problem is to begin by solving three different problems. Each problem is the same differential equation as the original, but each has only one inhomogeneous boundary condition. In the first such sub-problem, we take $V = V_T$ on top, but $V = 0$ on the side and bottom. What are the other two sub-problems?

3. If you solved all three sub-problems and added the solutions, would the resulting function solve the original differential equation? Would it solve all the original boundary conditions?