Discovery Exercise for The Method of Eigenfunction Expansion

The temperature in a bar obeys the heat equation, with its ends fixed at zero temperature:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$u(0,t) = u(L,t) = 0$$

You can find the temperature $u(x,t)$ of such a bar by using separation of variables; you are now going to solve the same problem (and hopefully get the same answer!) using a different technique, "eigenfunction expansion." The method of eigenfunction expansion can be used in some situations where separation of variables cannot—most notably in solving some inhomogeneous equations.

To begin with, replace the unknown function $u(x,t)$ with its unknown Fourier sine expansion in x.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right) \tag{2}$$

1. Does this particular choice (a Fourier series with no cosines) guarantee that you meet your boundary conditions? If so, explain why. If not, explain what further steps will be taken later to meet them.

2. What is the second derivative with respect to x of $b_n(t)\sin(n\pi x/L)$?

3. What is the first derivative with respect to t of $b_n(t)\sin(n\pi x/L)$?

4. Replacing $u(x,t)$ with its Fourier sine series as shown in Equation 2, rewrite Equation 1.

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Now we use one of the key mathematical facts that makes this technique work: if two Fourier sine series with the same frequencies are equal to each other, then the coefficients must equal each other. For instance, the coefficient of $\sin(3x)$ in the first series must equal the coefficient of $\sin(3x)$ in the second series, and so on.

5. Set the nth coefficient on the left side of your answer to Part 4 equal to the nth coefficient on the right. The result should be an *ordinary* differential equation for the function $b_n(t)$.

6. Solve your equation to find the function $b_n(t)$.

7. Write the function $u(x,t)$ as a Fourier sine series, with the coefficients properly filled in.

8. How will your temperature function behave after a long time? Answer this question based on your answer to Part 7; then explain why this answer makes sense in light of the physical situation.