Discovery Exercise for Lagrange Multipliers

There are two functions in this exercise, and it’s important not to confuse them.

- The “objective” function, \( f(x, y) \), is the one we really care about. This function is not shown in the drawing below.
- The other function will be called here \( g(x, y) \). When we set this function equal to a constant we get the curve \( g(x, y) = k \) shown below. That curve is our “constraint.”

We are interested here in values of the function \( f(x, y) \), but only along the curve defined by \( g(x, y) = k \). Specifically we are interested in finding the maximum value that \( f(x, y) \) attains along that curve. Note that this may not correspond to a local maximum of the function \( f(x, y) \).

The drawing shows the curve \( g(x, y) = k \) and three points \( P_{\text{left}} \), \( P \), and \( P_{\text{right}} \) on that curve. The vector \( \vec{v} \) points parallel to the curve at position \( P \), generally in the direction of \( P_{\text{right}} \).

1. For this part only, suppose that \( D_{\vec{v}} f \) at point \( P \) is positive.
   
   (a) As you move from \( P \) toward \( P_{\text{right}} \) does the value of \( f(x, y) \) increase, decrease, or stay the same?
   
   (b) As you move from \( P \) toward \( P_{\text{left}} \) does the value of \( f(x, y) \) increase, decrease, or stay the same?

2. Now, for this part only, suppose that \( D_{\vec{v}} f \) at point \( P \) is negative. Explain how we know that point \( P \) cannot possibly represent the maximum value of \( f \) along the curve.
For the remaining questions in this exercise, suppose that point $P$ does in fact represent the maximum value of $f$ along the curve.

3. What does that assumption imply about $D_v f$ at point $P$? Explain briefly how you know.

4. What does your answer to Part 3 imply about the gradient $\nabla f$ at point $P$? Explain briefly how you know.  
   \textit{Hint:} it doesn’t imply $\nabla f = 0$.

5. Which way does $\nabla g$ point at point $P$? Explain briefly how you know.

6. Use your answers to Parts 4–5 to write an equation relating $\nabla f$ and $\nabla g$ at the point $P$ where $f$ takes on its maximum along the curve $g = k$. 