Discovery Exercise for the Gradient

Consider a function $f(x, y)$ with constant partial derivatives $\partial f/\partial x = 1$ and $\partial f/\partial y = 2$.

1. Find the directional derivative of f in the direction $\hat{i} + c\hat{j}$. Your answer will depend on the unknown constant c.

   See Check Yourself #23 at felderbooks.com/checkyourself

2. Find the value of c that maximizes the directional derivative of f.

3. In what direction does f increase the fastest? Give your answer in the form of a vector in the xy plane that points in the direction of fastest increase of f.

4. Normalize your answer to give a unit vector $\hat{u}$ that points in the direction of fastest increase of f.

Now we'll generalize the problem to a function $f(x, y)$ for which $\partial f/\partial x = a$ and $\partial f/\partial y = b$.

5. Repeat the steps above. Find the directional derivative of f in the direction $\hat{i} + c\hat{j}$ and use that to find the direction in which f increases the fastest. Once again, express your final answer as a unit vector $\hat{u}$ that points in the direction of fastest increase of f.

6. What is the rate of change of f in the direction you just found? In other words, what is the fastest rate of increase that f can have in any direction?

7. Multiply the vector you found in Part 5 by the answer you found in Part 6.

Your final result was a vector that points in the direction in which f increases the fastest and whose magnitude is the rate of change of f in that direction. That vector is called the "gradient" of f.