Computer Problems for Partial Derivatives

1. Planck's Law of blackbody radiation states that a completely black object will emit radiation according to the formula:

$$I(\nu, T) = \frac{2h\nu^3}{c^2\left(e^{\frac{h\nu}{kT}} - 1\right)}$$

where I is radiated power, ν is the frequency of emitted radiation, T is temperature in Kelvin, and h, c, and k are positive constants.

   (a) Calculate $\partial I/\partial T$. Is it positive or negative? What does this tell you about the blackbody radiation emitted by different objects?

   (b) Calculate $\partial I/\partial \nu$.

   (c) Pick any three values for the ratio $h/(kT)$ and plot $\partial I/\partial \nu$ for each of those values. (You may assume you're working in units where $h = c = 1$.)

   (d) The three plots should all have the same basic shape. Describe this shape and explain what it tells you about the blackbody radiation emitted by an object.

   (e) For each plot there should be a positive value $\nu = \nu^*$ for which $\partial I/\partial \nu = 0$. Estimate that value for each temperature. How does ν^* depends on temperature? What does that tell you physically about the blackbody radiation emitted by an object?

2. A differential equation is an equation involving derivatives. If it involves partial derivatives, it's called a "partial differential equation." Plug each of the given solutions into the partial differential equation $\partial f/\partial t = \left(1 - x^2\right)\left(\partial^2 f/\partial x^2\right) - 2x(\partial f/\partial x)$ to see whether it's a valid solution. More than one of the solutions may work.

   Note that several of the proposed solutions below involve functions that you may not have heard of, but you can still use a computer algebra program to take their derivatives and check them in the given equation.

   (a) $f(x, t) = \sin x e^{-2t}$

   (b) $f(x, t) = J_1(x)e^{-2t}$ ("Bessel function")

   (c) $f(x, t) = P_1(x)e^{-2t}$ ("Legendre polynomial")

   (d) $f(x, t) = Ai(x)e^{-2t}$ ("Airy function")

   (e) $f(x, t) = H_1(x)e^{-2t}$ ("Hermite polynomial")

3. A curve is defined by the relation $y^3 + y = e^x + x$.

   (a) Use implicit differentiation to find the slope of this curve. Call the resulting function $m(x, y)$.

   (b) Is $m(x, y)$ positive or negative? Explain in words what this tells us about the shape of the curve.

   (c) Plot $y^3 + y = e^x + x$ on a computer and confirm that it matches you said in Part (b).

4. A scalar field is given by $f(x, y) = e^{-(x+1)^2 - y^2} + e^{-(x-1)^2 - y^2}$.

   (a) Calculate the gradient $\vec{\nabla} f$.

   (b) Have a computer generate a 3D plot of the field with x and y on the horizontal axes and f on the vertical axis.

   (c) Have the computer generate either a contour plot of f (showing lines of constant f) or a density plot (shading regions of the plot according to the value of f). This plot will be 2D, with only x and y axes.

   (d) Have the computer plot the vectors $\vec{\nabla} f$ on the same plot as your contour or density plot.

   (e) Explain why the image you generated looks the way it does. In other words, looking at the contour lines or shading, how could you have predicted what the gradient vectors would be doing?

5. Generate a plot of the function $z = \sin(x^2 y)$ in the range $-2 \le x \le 2$, $-2 \le y \le 2$ and of its power series at different orders. What order do you need to go to before the power series plot looks nearly identical to the plot of the actual function?

6. Find all critical points of the function $x^4 + 2x^3 - x + y^2 + z^2 + xyz$. Use the second derivatives test to classify each critical point.

7. Find the point on the paraboloid $z = x^2 + y^2$ that is closest to the point $(1, 2, 0)$. To put it another way, your job is to find the point with the minimum distance to $(1, 2, 0)$, subject to the constraint that your answer must lie on this paraboloid. *Hint:* Rather than minimizing "distance to P" it may be easier to minimize "distance to P squared," which will give you the same point.

Problems 8–9 are about the "least squares method" for finding a function $y_{fit}(x)$ that approximates ("fits") a set of data points. A perfect fit would be one that exactly went through every data point, but that usually makes the fitting functions too complicated. To measure the accuracy of the fit you measure the vertical distance from each point to the curve, $y_{data}(x) - y_{fit}(x)$, square them, and add them together to get the "squared error." For example, if you fit the points $(1, 1)$, $(2, 7)$, $(3, 30)$ with the curve $y_{fit} = x^3$ then the squared error would be $(1-1)^2+(7-8)^2+(30-27)^2 = 10$. The smaller the squared error is, the better the fit. (Squaring is important because it means you are counting every error as positive whether the data point is above or below the line.)

8. Consider the data shown below.

x	1	2	3	4	5
y	1.4	2.7	7.2	5.6	8.1

   (a) Find the best-fit cubic function $y_{fit} = ax^3 + bx^2 + cx + d$ to these data.

   (b) Find the best-fit cubic function $y_{fit} = ax^3+bx^2+cx+d$ to these data subject to the constraint $a+b+c+d = 1$.

   (c) Plot the original data points and the two fitting curves you found together on the same plot.

9. In this problem you will see how a moderately large data set can be fit by different types of curves. (In practice engineers optimize functions with as many as several thousand variables, but this one is at least large enough that you wouldn't want to do it by hand.) Your data set will consist of 21 evenly spaced points from $x = 0$ to $x = \pi$ on the plot of $y = \cos x$: $(0, 1)$, $(\pi/20, \cos(\pi/20))$, $(2\pi/20, \cos(2\pi/20))$, $(3\pi/20, \cos(3\pi/20))$, $\ldots(\pi, -1)$.

   (a) Have the computer calculate the squared error for a linear fit $y_{fit} = mx + b$ to these data points. Your answer should be a function of m and b.

   (b) Have the computer minimize that function of m and b (either by setting the partial derivatives equal to zero or by using the program's built-in minimization function). The result should be a best-fit line y_{fit}. What is the squared error for this fit?

   (c) Plot the data and the line together on one plot.

   (d) Find the best-fit quadratic function for these data. Give the squared error and plot the data and the fitting function together on one plot.

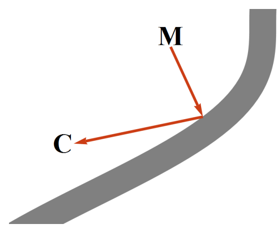
   (e) Find the best-fit cubic function for these data. Give the squared error and plot the data and the fitting function together on one plot.

10. Coulomb's law says that a point charge q produces an electric field $\vec{E}$ whose magnitude at each point equals $|q|/(4\pi\varepsilon_0 r^2)$, where ε_0 is a constant and r is the distance from the charge. The direction of $\vec{E}$ is away from the charge if $q > 0$ and toward it if $q < 0$. When you have more than one charge the electric field at any given point is the *vector* sum of the fields produced by each charge, so you have to break the field from each charge into components and then add them. Suppose a charge $q = 5$ is placed at the point $(0, 1)$. Your goal is to place two negative charges $q = -3$ and $q = -4$ somewhere on the the unit circle so as to minimize the magnitude of $\vec{E}$ at the origin.

    (a) Using Lagrange multipliers write, but do not yet solve, the six equations for the six unknowns (the x and y positions of the two particles, plus two Lagrange multipliers for the two constraints).

    (b) Solve your equations to find the positions of the two negative charges.

11. There is a classic optimization problem known as the "milkmaid problem." A milkmaid is at the location (x_M, y_M) and she needs to milk a cow at location (x_C, y_C). Before milking the cow, however, she needs to stop by the river to wash her pail.



The path of the river is described by the equation $g(x, y) = 0$ for some function g. Her goal is to find the point (x, y) on the river that will allow her to wash her pail and then get to the cow with as little travel distance as possible.

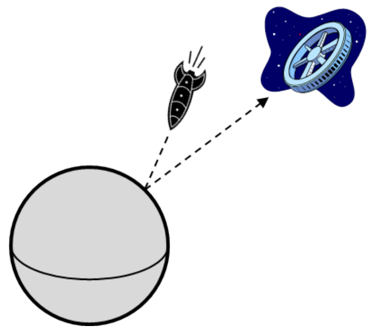
    (a) Using Lagrange multipliers, write the equations that need to be solved to find the optimal point (x, y) where the milkmaid should wash her pail.

    (b) Write and solve those equations assuming the river flows long the x-axis, the milkmaid's initial position is $(3, 2)$, and the cow's position is $(1, 2)$. Explain why your answer makes sense.

    (c) Write and solve those equations assuming the river flows long the line $y = x$, the milkmaid's initial position is $(1, 3)$, and the cow's position is $(3, 5)$. Explain why your answer makes sense.

    (d) Assume the river is described by the curve $y = x^3$, the milkmaid's initial position is $(0, 2)$, and the cow's position is $(1, 4)$. Have a computer numerically calculate where the milkmaid should wash her pail. Make a plot showing the river, the two points given here, and the point you just found.

12. Your spaceship needs to gather rock samples from a planet's surface and deliver them to a nearby space station. You are currently at location $(50, 10, 20)$ and the space station is at $(30, 20, 30)$. (All coordinates are measured in thousands of miles in a coordinate system with the origin at the center of the planet.) You need to choose where to gather the rocks so that you can reach the space station with as little distance travelled as possible.



(a) Write the function you are trying to minimize in terms of the coordinates (x, y, z) of the point where you will gather the rocks.

(b) The planet's surface is a sphere of radius 10 centered on the origin. Write the constraint equation that x, y, and z must satisfy.

(c) Set up, but do not yet solve, the equations to find the coordinates of the point where you should gather the rocks.

(d)  Numerically solve the equations you wrote to find the coordinates where you should land on the planet.