Discovery Exercise for Solving Differential Equations with Power Series

In this exercise you are going to find a particular solution to the following differential equation. (You may already know how to solve this equation, but our purpose here is to demonstrate a technique.)

$$\frac{dy}{dx} = y + \sin x \tag{1}$$

Our goal is to find a function $y(x)$ that satisfies Equation 1. Now, suppose we found such a function and then expanded it out into a Maclaurin series. Then it would look like this.

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots \tag{2}$$

So instead of saying "we need to find the mystery function" we can reframe the question as "we need to find the mystery coefficients" (which, in turn, define the function itself).

1. Based on Equation 2 write a formula for dy/dx.

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Just to save you a bit of time, we will tell you—although we hope you could rapidly figure this out yourself, or possibly even have it memorized, that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

2. Plug Equation 2, and your answer to Part 1, and the Maclaurin expansion of $\sin x$, into Equation 1.

3. Combine like powers of x on the right side of your answer to Part 2.

4. Your equation in Part 3 has a constant term on the left and a constant term on the right. Write an equation asserting that these two must be equal.

5. Your equation in Part 3 has a coefficient of x on the left and a coefficient of x on the right. Write an equation asserting that these two must be equal.

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6. In like manner, write equations setting the coefficients of x^2, x^3, and x^4 equal to each other. You now have five equations relating the coefficients.

7. Choose $c_0 = 0$. Based on the equations you have written, find all the other coefficients up to c_5.

8. Based on your coefficients, write down a solution to Equation 1. It will not be an exact answer, but it will be the power series of the solution up to the fifth term. *Hint*: Several of the terms will be zero, and some won't.