Computer Problems for Special Functions and ODE Series Solutions

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

1. The function $y = \ln x$ cannot be expanded out into a Maclaurin series (why?), but it can be expanded into a Taylor series around $x = 1$. Beginning with $y = \sum c_n (x-1)^n$ find the particular solution, up to the fourth power, of the equation $4y'' + y = \ln x$ with conditions $y(1) = 8$, $y'(1) = 1$.

2. *[This problem depends on Problem 1.]*

   (a) Have a computer get the exact solution to $4y'' + y = \ln x$ with initial conditions $y(1) = 0$, $y'(1) = 1$. Comment on why a power series approach might be better suited to this problem, even though there is an exact solution.

   (b) Plot the fourth-order Taylor series solution that you found in Problem 1 and the exact solution together on the same plot, starting at $t = 0$. Experiment with the final value of t until you can see at what value the two solutions start to diverge significantly from each other. Estimate this value.

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3. Have a computer calculate the Legendre series for $f(x) = x^{-1/3}$ up to the eleventh order. Plot that partial sum and the function $f(x)$ on the same plot from $x = -2$ to $x = 2$. Describe how the Legendre series approximates the function near $x = 0$, in the region $0 < |x| < 1$, and in the region $|x| > 1$.

4. In this problem you're going to compare the Legendre series and the Maclaurin series for the function $f(x) = \cos(2x)$.

   (a) Calculate the second-order Maclaurin series for $\cos(2x)$.

   (b) Calculate the second-order Legendre series for $\cos(2x)$. Expand the Legendre polynomials in your answer and gather like powers of x so it's in the form of a quadratic function of x, just like the Maclaurin series.

   (c) On one plot, show $\cos(2x)$ and the two quadratic approximations you just calculated from $x = -0.1$ to $x = 0.1$. Which quadratic better approximates the function in this interval?

   (d) On one plot, show $\cos(2x)$ and the two quadratic approximations you just calculated from $x = -1$ to $x = 1$. Which quadratic better approximates the function in this interval?

   (e) Taylor series and Legendre series are two different ways of approximating a function with a polynomial. Based on your results, what are Taylor series more useful for and what are Legendre series more useful for?

5. Calculating the "norm" of a set of functions is a key step in finding the formula for the coefficients of a series expansion. In this problem you will use Rodrigues' formula to derive the norm of the Legendre polynomials: $\int_{-1}^{1} [P_l(x)]^2 dx$.

   (a) Substitute for $P_l(x)$ using Rodrigues' formula. Pull the constant out of the integral.

   (b) Integrate by parts and argue that the term outside the integral must equal zero.

   (c) Find the integral you're left with after l integrations by parts.

   (d) That integral involves the $(2l)^{\text{th}}$ derivative of $(x^2-1)^l$. Evaluate that derivative. *Hint*: if you expand $(x^2-1)^l$ all but one of the terms will vanish when you take the derivative.

   (e) Evaluate the integral and compute the norm of the Legendre polynomials.

   (f) The norm of the Legendre polynomials is most simply expressed as $\int_{-1}^{1} [P_l(x)]^2 dx = 2/(2l+1)$. Your computer may have given you an answer that looks different from this, but you can check it by calculating some of the terms. Calculate $\int_{-1}^{1} [P_l(x)]^2 dx$ for $l = 1$, 2, and 3 using the answer you got and, if it looks different, the formula we just gave you. Make sure they match.

6. The parts of this problem refer to "Bessel functions of the first kind" $J_p(x)$ and "Bessel functions of the second kind" $Y_p(x)$. Refer to your software's documentation to find the proper syntax for entering these functions. For all the following questions, use the domain $0 \leq x \leq 50$.

   (a) Graph the function $J_1(x)$. Answer in words: how is $J_1(x)$ like a sine function? How does it differ from a sine function?

   (b) How many zeros does $J_1(x)$ have in this domain?

   (c) The first three positive zeros of this function are called $\alpha_{1,1}$, $\alpha_{1,2}$, and $\alpha_{1,3}$. Find their values. Your answers should be accurate to the second decimal place.

   (d) Find the absolute maximum of $J_1(x)$.

   (e) Graph the functions $J_{1.2}(x)$, $J_2(x)$, and $J_{15}(x)$. Answer in words: how are these functions alike? How do they differ?

   (f) Graph the function $Y_1(x)$. Answer in words: how is this function like $J_1(x)$? How is it different?

7. $J_p(x)$ and $J_{-p}(x)$ are linearly dependent for integer values of p. The function $Y_p(x)$ is therefore constructed to be the second solution to Bessel's equation. In this problem you will show for a few specific cases that $J_p(x)$ and $Y_p(x)$ are linearly independent.

   (a) Choose two positive numbers, one integer and the other non-integer. Plot $J_p(x)$ and $Y_p(x)$ for $p = 0$ and for p equal to each of the two numbers you chose, six plots in all. Choose domains for the plots that allow you to see the behavior for both small and large positive values of x. (You do not need to include negative values of x in your plot.)

   (b) Based on your plots, describe how each of these six functions behaves in the limit $x \to 0^+$.

   (c) Based on your answer to Part (b) argue that J_p and Y_p cannot be linearly dependent for the three values of p you considered.

8. In this problem you'll examine the behavior of $J_{.1}$, $J_{.5}$, and $J_{1.5}$.

   (a) Plot all three functions from $x = 0$ to $x = 100$. Describe the plots. What is happening to the amplitude? To the frequency?

   (b) For each function, make a list of the first 100 zeroes. Then, for each function, make a list of the differences between successive zeroes. (For example, if you were doing this for $\sin x$ the second list would be $(\pi, \pi, \pi, \ldots)$. Describe what is happening to the distance between zeroes in each case.

   (c) Repeat Parts (a)–(b) for $Y_{.1}$, $Y_{.5}$, and $Y_{1.5}$.

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9. In this problem you'll derive the coefficients for a zero-order Fourier-Bessel series. In other words, you'll figure out how to write an arbitrary function $f(x)$ on the interval $[0,1]$ as a sum of the functions $J_0(\alpha_{n,0}x)$, where J_0 is a "zero-order Bessel function" and $\alpha_{0,n}$ is one of the zeroes of J_0 (meaning $J_0(\alpha_{n,0}) = 0$). The best part is, you could do all this even if you'd never heard of a Bessel function. Your starting point is the orthogonality relation $\int_0^1 xJ_0(\alpha_{0,m}x)J_0(\alpha_{0,n}x)dx = (1/2)J_1^2(\alpha_{0,m})\delta_{mn}$. (We're not going to prove that here, but Sturm-Liouville theory promises us that such a relationship must exist.)

   (a) What is the weight function in this orthogonality relationship?

   (b) Your goal is to find the coefficients of the series $f(x) = \sum_{n=1}^{\infty} c_n J_0(\alpha_{0,n}x)$. Start by multiplying both sides of this equation by $xJ_0(\alpha_{0,m}x)$.

   (c) Integrate both sides of the equation from $x = 0$ to $x = 1$. You should be able to use the orthogonality relation to eliminate all but one of the terms in the infinite sum.

   (d) Solve for the coefficient c_m. Your answer should include an integral that involves the unknown function $f(x)$.

10. *[This problem depends on Problem 9.]* Find the first three coefficients of the zero-order Fourier-Bessel series for $f(x) = x$ on the interval $[0,1]$. Use a computer to evaluate the necessary integrals numerically.

11. **Exploration: A Different Kind of Series Solution**

The following differential equation describes the motion of an object falling in the gravitational field of a planet.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{1}$$

Here r is the distance of the object from the center of the planet. The given constants in this problem are G (a universal constant), M (the mass of the planet, not of the falling object), and r_0 and v_0 (the initial position and velocity of the object).

(a) Explain why this equation does not lend itself directly to either the power series method, or the method of Frobenius.

Nonetheless, we can approach this problem by looking for the first few coefficients in a Maclaurin series solution:

$$r(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \ldots \tag{2}$$

(b) Using Equation 2 find the first two coefficients in terms of the given constants.

(c) Take the second derivative of both sides of Equation 2. Then use Equation 1 to replace d^2r/dt^2, and finally plug 0 into both sides to find c_2 in terms of the given constants.

(d) Write the solution to Equation 1 up to the second order. This solution should look like the introductory mechanics equation $x = x_0 + v_0 t - (1/2)gt^2$ with the constant g being a function of our given constants. (This g should come out as 9.8 m/s^2 if you use the mass and radius of the Earth as M and r_0.)

(e) To find the next term—the first correction to the introductory mechanics equation—take the derivative of both sides of Equation 1 with respect to time, and then plug in $t = 0$. Solve for c_3 in terms of the given constants.

(f) Write the solution to Equation 1 up to the third order.

(g) We have seen that your second order formula replicates the introductory mechanics equation $r = r_0 + v_0 t - (1/2)gt^2$, which works well for objects that stay near the surface of the Earth ($r \approx R$). Does the third order correction make the effective acceleration due to gravity higher, or lower, than 9.8 m/s^2? Answer based on your equation, but then explain why your answer makes sense physically. Assume $r_0 = R$ and $v_0 > 0$ (such as a rocket taking off). (Your answer will depend on time.)

(h) A bullet is fired straight up into the air from the surface of the Earth with an initial speed of 1000 m/s. Look up values for G, M, and r_0, and use them to graph the height of the bullet using your answers to Parts (d) and (f). How does adding the third-order term change the motion of the bullet?