Motivating Exercise for Ordinary Differential Equations: The Simple Harmonic Oscillator

Figure 1 shows an object of mass m attached to a spring. The spring obeys "Hooke's Law" which states that $F = -kx$, where...

- F is the force the spring exerts on the object.
- x is the the object's displacement from its relaxed position. Note that x can be positive (if the spring is stretched) or negative (if the spring is compressed).
- k is a positive constant, the "spring constant."

In SI units F would be measured in Newtons and x in meters, and k would therefore be in N/m.

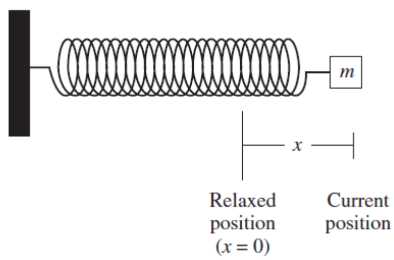

Figure 1

1. What does Hooke's Law predict about the *direction* of the force the spring exerts? Based on this, what kind of motion would you expect?

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Newton's Second Law tells us that $F = ma$; the force exerted by the spring on the object determines the acceleration of the object, where acceleration is the second derivative of position with respect to time.

2. Using Hooke's Law and Newton's Second Law, write an equation that relates the position of the mass x to its acceleration a. Your equation will have two constants in it: m (the mass of the object) and k (the spring constant).

3. Replace a with its definition d^2x/dt^2 and rewrite your equation so that the second derivative is on the left side of the equation by itself.

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4. One trivial solution to that equation is the simple function $x(t) = 0$. (If you take the second derivative, you do indeed get the same function back, times anything you like.) What does this solution tell us about one possible motion of the mass-on-spring system?

5. In order to find other possibilities, let's begin by supposing $k/m = 1$, so $d^2x/dt^2 = -x$. Which of the following functions is *the negative of its own second derivative*? One and only one of these functions works. (These solutions all appear to have incorrect units, but by setting $k/m = 1$ we've hidden the units. They will reappear in Part 6.)

   (a) $x = -1$
   (b) $x = -(1/6)t^3$
   (c) $x = e^t$
   (d) $x = e^{-t}$
   (e) $x = \sin t$

6. Now let's return to the original differential equation: $d^2x/dt^2 = -(k/m)x$. To be a solution to this equation it should work no matter what the values of k and m are. Which of the following functions is a solution to this equation? For each one, you need to test it by taking its second derivative and checking whether the answer you get is $-k/m$ times the original function. Once again, one and only one of these works.

   (a) $x = \sin t$
   (b) $x = \sin(kt/m)$
   (c) $x = \sin\left(\sqrt{k/m}\; t\right)$
   (d) $x = \sin(mt/k)$
   (e) $x = \sin\left(\sqrt{m/k}\; t\right)$

7. The function that you just found as a solution to $d^2x/dt^2 = -(k/m)x$ offers a prediction for the motion of the mass. Does this prediction correspond to the prediction you made on physical grounds in Part 1?