Motivating Exercise for Introduction to Differential Equations Flow of a Compressible Fluid

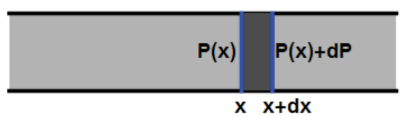
The Hagen-Poiseuille equation describes laminar flow of an incompressible fluid through a circular pipe of constant radius r. Taking a horizontal pipe for simplicity (hence no effects from gravity) the equation says the following.

$$w = -\frac{\pi r^4}{8\mu}\rho\frac{\Delta P}{L}$$

Here w is the mass flow rate (mass of fluid passing any point in the pipe per unit time), r is the radius of the pipe, μ is the viscosity of the fluid, ρ is the density of the fluid, L is the length of the pipe, and ΔP is the change in pressure from the beginning of the pipe to the end.

Now consider an ideal gas at constant temperature T flowing steadily through a circular pipe. Ideal gases are *not* incompressible; their density is related to their pressure by the ideal gas equation, which can be written as $P = \rho RT/M$, where M is the molar mass of the fluid.[2]

The Hagen-Poiseuille equation applies to incompressible fluids—that is, it assumes constant density. To apply it to a compressible fluid like an ideal gas, we employ a common strategy: consider a segment of pipe so short that we can treat the density as constant throughout that length and apply the Hagen-Poiseuille equation to that segment.



1. Consider a short segment of length dx in the middle of the pipe, shown above. Call the pressure drop across that small segment dP and write the Hagen-Poiseuille equation for that short length of pipe.

2. Your equation should have ρ in it. Use the ideal gas equation to eliminate ρ and get an expression for w that only depends on P, dP, dx, and constants.

3. Rewrite your equation so that dP/dx is by itself on the left side.

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[2]You can write that in the possibly more familiar form $PV = nRT$ if you recognize that the density ρ equals nM/V, i.e. number of moles times mass per mole, divided by volume.

What you've just derived is a "differential equation" for the function $P(x)$, specifically one that says that the derivative of $P(x)$ is equal to a constant divided by $P(x)$. At this point you may have no idea how you would go about finding such a function, so we're just going to give you one.

$$P(x) = \sqrt{P_0^2 - \frac{16\mu RTw}{\pi r^4 M} x} \tag{1.3}$$

Here P_0 is the pressure at the point $x = 0$.

4. Calculate dP/dx using Equation 1.3. Remember that all the letters except x are constants.

5. Plug $P(x)$ from Equation 1.3 and the dP/dx you just calculated into the differential equation you wrote above and verify that it works, meaning both sides of the equation come out equal.

The good news is that you just used a differential equation to find an expression for the pressure $P(x)$ for an ideal gas flowing through a pipe. The bad news is that you might have no confidence that you could do a similar process to derive the differential equation for a different situation, and we didn't give you any clue how we figured out that solution. Those are both skills that you will learn as you study differential equations.