Motivating Exercise for Integrals in Two or More Dimensions: Newton's Problem (or) The Gravitational Field of a Sphere

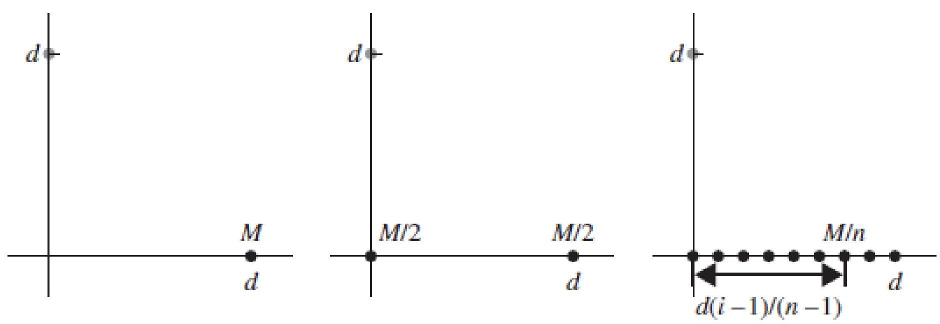
Isaac Newton had a problem. His new law of universal gravitation did a great job of explaining apples falling from trees. It also accounted perfectly for Kepler's observationally-based laws of planetary motion, as long as he assumed that the planets, the sun, and the moon were all point particles. The problem was, they weren't. Newton needed to show that a full sphere would *act like* a point particle located at its center.

In this exercise you will take the first steps toward the proof that Newton needed. (Spheres will not come into this exercise at all, but journey to Newton's problem begins here.) The only physics you need to know is that a particle with mass M creates a "gravitational potential" at every point around it, given by the formula:

$$V(x, y, z) = -GM/r \tag{1}$$

where G is a constant and r is the distance from the particle to the point. If there is more than one particle around, the gravitational potential at each point is the sum of the potentials created by all the nearby particles. (Gravitational potential is used to figure out gravitational force, but we won't be doing that here.)

In all the problems below, the question is the same: *Use Equation 1 to calculate the gravitational potential at the point* $(0, d)$ *on the* xy *plane.*



1. Suppose the only object around is a particle of mass M at the point $(d, 0)$ (the first image above). Find the gravitational potential.

2. Now suppose there are two objects, each of mass $M/2$, located at $(0, 0)$ and $(d, 0)$. (See the second image.) Calculate the gravitational potential.

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3. Now suppose there are n objects, each of mass M/n, spread evenly between $(0,0)$ and $(d,0)$ (third image). Object number i is located at position $x_i = d(i-1)/(n-1)$.

   (a) Check that this gives the correct positions for x_1 and x_n.

   (b) Calculate the gravitational potential created by object i.

   (c) Write a sum representing the total potential at $(0,d)$ created by this collection of objects. (You do not need to evaluate the sum; just write it.)

In the limit as $n \to \infty$ your final answer becomes the total gravitational potential generated by a mass M spread evenly between $x=0$ and $x=d$. But calculating such a quantity by adding up a series and then taking a limit is generally not practical. The most important step to understanding integrals is learning to think of them as the solution to *that problem:* a fast method for calculating the limit of the sum.

Setting up and solving problems like this is the most important skill in integral Calculus—indeed, one of the most important skills in mathematics. As you study multivariate Calculus you will see how to extend such an integral into two or three dimensions, and then how to integrate in curvilinear coordinate systems. With these tools in hand you can return to, and solve, Newton's problem.