Chapter: Integrals in Two or More Dimensions

Discovery Exercise for Line Integrals

We begin with a fact from physics: if a force \( \vec{F} \) acts upon an object that moves through a linear displacement \( d\vec{s} \), the work done by the force on the object is given by the dot product \( W = \vec{F} \cdot d\vec{s} \).

1. Calculate the work done by a force \( \vec{F} = 2\hat{i} + 3\hat{j} \) on an object as it moves along a line from the point \((1, 1)\) to \((5, 25)\).

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Part 1 does not require an integral, because a constant force is acting along a straight line. If the force or the path varies, you employ the usual strategy: break the journey into small parts, compute each part with a simple dot product like Part 1, and use an integral to add up all the contributions.

Consider, for instance, a bead moving along a wire in the shape of the curve \( y = x^2 \). A force \( \vec{F} = (x + y)\hat{i} + (xy)\hat{j} \) acts on the bead as it moves from \((1, 1)\) to \((5, 25)\). Your job in this exercise is to calculate the work done by the force on the bead.

We begin with a small part of the journey—small enough to treat the force as a constant and the journey as a line.

2. The force is expressed as a function of \( x \) and \( y \). Re-express the force—both its \( x \) and \( y \) components—as a function of \( x \) only. Remember what curve we are moving along!

3. The figure above represents a small displacement along the curve \( d\vec{s} = dx \hat{i} + dy \hat{j} \). Express the relationship between \( dy \) and \( dx \), based once again on the curve. (Hint: It’s not \( dy = dx^2 \).)

4. Calculate the work done along this small displacement. Your answer should be expressed as a function of \( x \) and \( dx \) only.

5. Integrate your answer as \( x \) goes from 1 to 5.

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