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-  A uniform rod of charge Q goes from the origin to the point $(d, 0)$. Find the electric potential produced by this rod at the point $(0, h)$. Note that you will need to use a picture to express the distance from a slice to the point in terms of x .
- A right circular cone of height H that makes a 45° angle with the horizontal has a density $a/(H^3 + z^3)$ where z is the distance from the ground.
 - Find the mass of the cone.
 -  Find the height of the center of mass of the cone.

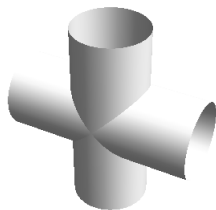
For Problems 3–4 find the center of mass of the given object, and find its moment of inertia about the z axis. For the center of mass you need to specify all of its coordinates, but you can often identify some (or all) of them by symmetry without any calculations. For example, a uniform cone centered on the z -axis has $x_{COM} = y_{COM} = 0$, but z_{COM} requires an integral.





-  A disk in the xz plane defined by $x^2 + z^2 \leq R^2$, with mass density $\sigma = kx^2$
 -  A uniform solid sphere of mass M and radius R centered on the origin.
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- Find the volume under the surface $z = e^{y/x}$ on $x \in [1, 3]$, $y \in [0, 4]$.
 - Write (but do not evaluate) the double integral such that the integral with respect to x should be taken first.
 - Write (but do not evaluate) the double integral such that the integral with respect to y should be taken first.
 - One of the two integrals can easily be evaluated by hand; the other cannot. Based on this, choose whether to use the form from Part (a) or Part (b).
 - Evaluate the first (inner) integral. The result should be a single integral that is impossible to evaluate by hand.
 -  Evaluate the second (outer) integral using a computer or calculator.
 - A rectangular region extending from $(0, 0)$ to $(7, 10)$ is filled with a charge density (charge per unit area) $\sigma = kye^{xy}$ where k is a positive constant. Your job is to find the total charge.
 - Write the double integral with the x integral first and evaluate it.
 -  Write the double integral with the y integral first. Evaluating this is considerably harder; you can do the inner integral using integration by parts, but will still need a computer for the outer integral. Use one and verify that you get the same answer you got in Part (a).
 -  Find the electric potential at the origin produced by a uniform square of charge Q with corners at $(0, 0)$ and (L, L) .
 -  A square with corners at $(0, 0)$ and $(2, 2)$ has a charge density given by $\sigma = \sin(\pi x + 2\pi y)$, where all quantities are measured in SI units (in which $k = 9 \times 10^9$). Set up and numerically evaluate an integral for the electric potential at the origin.
 -  A two meter by two meter square of paper is colored such that the amount of ink per unit area is proportional to the distance from one of the corners. Write the density function and find the total amount of ink on the paper. (The answer will contain a constant of proportionality.)

10.  The region bounded by the curves $y = (x - 2)^2$ and $y = \sqrt{x}$ is filled with a charge density $\sigma = \sin(y)/x$. Find the total charge in this region.
- Sketch the two curves and the region between.
 - Find the x and y coordinates of the points of intersection of the two curves. (These will be essential for finding the limits of integration.) One of these points can be found just by thinking about it, but you will need a computer or calculator to find the other.
 - The easiest (and therefore best) way to set up this problem is with vertical slices. The lower limit on y is the same for all slices, and the upper limit on y is the same for all slices. Write the double integral this way and evaluate it numerically to find the total charge.
 - Horizontal slices in this case require breaking the region up into two subregions. In the lower region, each slice is bounded on the left by $x = 2 - \sqrt{y}$ and on the right by $x = 2 + \sqrt{y}$. In the upper region, each slice is bounded on the left by $x = y^2$ and on the right by $x = 2 + \sqrt{y}$. Find the total charge of these two regions separately, and add them to find the total charge. You should of course get the same answer you found with vertical slices!
11.  A uniform circle of plastic of radius R has total charge Q . Find the electric potential produced by the circle at a point on its edge.
12.  Find the moment of inertia of a uniform half-disk of mass M and radius R about its flat edge.
13.  Integrate the function $f = z$ in the region above the surface $z = x^2 + y^2$ and below the surface $x^2 + y^2 + z^2 = 2$.
14.  Find the volume of the region bounded by $z = x^2 + y^2$ and $z = 2x + 8$.
15.  A solid hemisphere of radius R sits with its flat end on the xy plane, centered at the origin. Its mass M and charge Q are uniformly distributed.
- Find the moment of inertia of this hemisphere about the z -axis.
 - Find the electric potential at the origin.
 - If we add another identical hemisphere underneath, we have a full sphere with mass $2M$ and charge $2Q$. What are its moment of inertia about the z -axis and potential at the origin?
16.  Find the electric potential created by a uniform cube of charge Q and length L at one of its corners.
17.  Find the integral of the function $f(x, y, z) = x^2 + z^2$ in the region between the surfaces $x^2 + z^2 = (y - 1)^2$ and $x^2 + z^2 = (y + 1)^2$.
18.  A uniform object of charge 3×10^{-9} fills an ellipsoid with semi-principal axis lengths 1, 2, and 3. Find the electric potential at the center of the ellipsoid. All numbers are given in SI units, in which $k = 9 \times 10^9$.
19.  Find the electric potential at a point $(0, 0, 5)$ due to a uniform sphere of radius 2 with charge 10^{-7} centered on the origin. All numbers are given in SI units, in which $k = 9 \times 10^9$.
20.  Integrate the function $f = 1/\rho$, where ρ is the polar angle, in the triangle with edges $x = 0$, $y = 0$, and $x + y = 1$.
21.  A uniform triangle with mass M lies in the xy plane with edges $y = 0$, $x = 3$, and $x = y$. Find the moment of inertia of this triangle about the z axis.



22.  A triangle lies in the xy plane with edges $x = 0$, $y = 0$, and $y = y_0 - 2x$. ($y_0 > 0$) The density of the triangle is proportional to distance from the z -axis. Find the moment of inertia of this triangle about the z axis. Your answer should contain a constant of proportionality from the density function. Be sure to check the units of that constant and use them to check the units of your answer.
23. A rectangle with corners at $(0, 0)$ and $(2\sqrt{3}, 2)$ is filled with a material of density $\sigma = 1/(x^2 + y^2 + 1)$.
- (a) Write a double integral to represent the mass of this region in Cartesian coordinates.
 - (b) Draw the rectangle, and draw in a diagonal line from $(0, 0)$ to $(2\sqrt{3}, 2)$. This diagonal line divides the rectangle into two triangles.
 - (c) Write a double integral to represent the mass of the upper triangle in polar coordinates.
 - (d) Write a double integral to represent the mass of the lower triangle in polar coordinates.
 - (e)  Compute the mass of this rectangle using both your Cartesian and your polar integrals. Make sure you get the same answer!
24. Region R is bounded by two parabolas, $y = 1/4 - x^2$ and $y = -1/4 + x^2$. An object fills region R with a constant density σ . Your job is to find the moment of inertia of this object around a perpendicular axis through the origin (the z -axis).
- (a) Draw region R .
 - (b) Set up the appropriate integral in Cartesian coordinates.
 - (c) Set up the appropriate integral in polar coordinates.
 - (d)  Evaluate whichever integral looks easier to find the answer.
25.  A cylinder of radius 1 m and height 2 m has a uniformly distributed charge 10^{-8} C. Find the electric potential produced by the cylinder at a point on the axis of the cylinder a distance 4 m from the center of the cylinder.


26. Imagine two right circular cylinders of radius 1, one centered on the x -axis and the other on the z -axis. A uniform object of mass 1 fills the space where the two cylinders intersect. You're going to find the moment of inertia of that object about the z -axis.



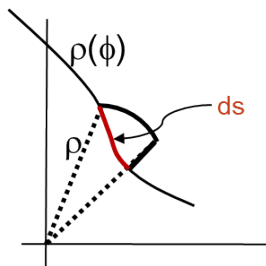
- (a) Explain why this problem is easiest to solve in cylindrical coordinates. (For the rest of the problem we'll use cylindrical coordinates where ρ is the distance from the z axis and ϕ is the angular variable.)
 - (b) Explain why it would be hard to set up the limits of integration for either the ϕ or ρ integrals first. You will therefore start with the z integral.
 - (c) We won't ask you to draw this shape with a box and a strip, but even without that you can see from the picture that a vertical strip will go from the bottom of the x -axis cylinder to the top. To turn that fact into limits of integration for z , write the formula for the x -axis cylinder *in Cartesian coordinates*.
 - (d) Now rewrite the formula for the x -axis cylinder in cylindrical coordinates, and use that to find the limits of integration for the z integral.
 - (e) With the z integral specified, the remaining 2D shape you need to integrate over in ρ and ϕ is the shadow of the 3D object on the xy -plane. Draw that shadow. (*Hint*: If you're stuck, think about the z -axis cylinder that goes up through this shape.)
 - (f) Using your 2D drawing, find the limits of integration for ϕ and ρ and write an expression for the moment of inertia of the object. Your expression should be a fraction with a triple integral in the numerator and in the denominator.
 - (g)  Evaluate your integral numerically to find I .
27. An object fills the ellipsoid $x^2 + y^2 + 2z^2 = 1$. Its density is given by the spherical $r^2 \sin \theta$. Find the mass of the object. (Throughout this problem r is the radial coordinate in spherical coordinates and θ and ϕ are the polar and azimuthal angles, respectively.)
- (a) What are the limits of integration for ϕ ?
 - (b) Draw the 2D shape that would rotate about the z -axis to make the ellipsoid. Explain why the horizontal axis in this drawing only needs a positive side.
 - (c) Rewrite the equation for the ellipsoid in spherical coordinates. After simplifying, your answer should be a function $r(\theta)$ with no ϕ dependence.
 - (d) Find the limits of integration for r and θ and write the triple integral for the mass of the object.
 - (e)  Evaluate your integral numerically to find M .
28.  Evaluate the line integral of $f(x, y) = x + y$ along the curve $y = x^3$ from $x = 0$ to $x = 2$.
29.  Evaluate the line integral of $f(x, y, z) = (x + y)/z$ along the curve $x(t) = t^2$, $y(t) = t^3$, $z(t) = 1/t$ from $(1, 1, 1)$ to $(9, 27, 1/3)$.

The next problem is not a computer problem but sets up the following two problems that do use computers.



30. Consider a single wire with current I .
- For a loop of radius ρ around the wire, the magnetic field \vec{B} has a constant magnitude. Calculate $\oint_C \vec{B} \cdot d\vec{s}$ around such a loop as function of $|\vec{B}|$ and ρ .
 - Use your answer and Ampere's Law to find $|\vec{B}|$ as a function of I .
31. *[This problem depends on Problem 30.]* In Problem 30 you found the field from a single wire carrying current I .
- Express your answer as a vector \vec{B} in unit vector notation.
 - Consider a circle of radius ρ surrounding the wire. Write functions $x(\phi)$ and $y(\phi)$ that parametrize that circle in terms of an angle and then use those functions to evaluate $\oint_C \vec{B} \cdot d\vec{s}$ around the circle. Your answer should confirm Ampere's law.
 - Still using the magnetic field around a single wire, set up the integrals you need to evaluate $\oint_C \vec{B} \cdot d\vec{s}$ around a rectangle that surrounds the wire. You'll have to choose where to put your axes and choose a specific rectangle to integrate around.
 -  Evaluate the integral you found in Part (c) and show that it obeys Ampere's law.
32.  *[This problem depends on Problem 30.]* A long, thin sheet of metal covers the region $-L \leq x \leq L$, $-\infty < y < \infty$. It is carrying a total current I in the positive y direction, with the current uniformly spread out across the width of the sheet.
- Using the result you found in Problem 30 for the field from a single wire, set up and evaluate a single integral (not a line integral) for the magnetic field produced at a point (x, y, z) by this sheet of metal.
 - Take the line integral of the magnetic field you found in Part (a) around the rectangle with corners at $(-2L, 0, -L)$ and $(2L, 0, L)$ and verify that it obeys Ampere's law.
 - Take the line integral of the magnetic field you found in Part (a) around a circle of radius $2L$ surrounding the sheet and verify that it also obeys Ampere's law.

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33.  A wire bent into the shape $z = 2x = y^2$ from $(2, 2, 4)$ to $(8, 4, 16)$ has linear density $\lambda = y/z$. Find the total mass of the wire.

34. Exploration: Scalar Line Integrals in Polar Coordinates

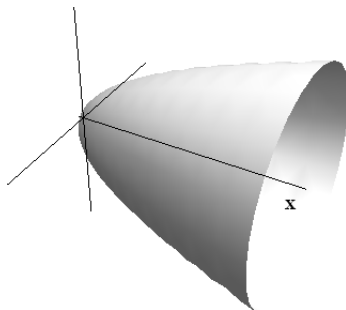



In Cartesian coordinates, we have seen that the differential ds can be written as $\sqrt{1 + (dy/dx)^2} dx$. This comes from a Pythagorean triangle where one leg is dx and the other is $(dy/dx)dx$. In polar coordinates we represent ds with a different Pythagorean triangle. We've drawn this "triangle" above, with the hypotenuse labeled ds . One leg is made by holding ρ constant while varying ϕ , and vice versa for the other.


- The "triangle" in the picture is not a real triangle because two of the sides are curved. Why is it still okay to treat it mathematically as if it were a triangle?
 - Label one leg in the drawing $d\rho$.
 - Label the other leg in the drawing. *Hint:* $d\phi$ is not the length of this leg, it is the angle it subtends. You want its length!
 - Use the Pythagorean theorem to find ds . Using the fact that the length $d\rho$ can be rewritten as $(d\rho/d\phi)d\phi$, express your answer in terms of ρ , ϕ , $d\rho/d\phi$, and $d\phi$. (These might not all show up in the answer.)
 -  Using the ds you just found, write and numerically evaluate a line integral in polar coordinates to find the arclength of the curve $\rho = \sin(3\phi)$ as ϕ goes from 0 to $\pi/3$.
35.  The equations $x = (a + b \cos v) \cos u$, $y = (a + b \cos v) \sin u$, $z = b \sin v$ where u and v both range from 0 to 2π define a torus. Graph this shape on a computer for different values of a and b . Describe in words what the shape looks like, and what a and b represent in the shape.

The next problem is not a computer problem but sets up the following problem that is.






36. The function $y = \sqrt{x}$ in the xy plane is rotated around the x -axis. In this problem you will write parametric functions for the resulting surface.





- (a) Explain why this surface cannot possibly be expressed by a function $z(x, y)$.
 - (b) Rather than u and v , the two parameters will be x (position on the x -axis) and θ (an angle rotating around the x -axis). Describe and/or draw the gridlines of constant x and constant θ .
 - (c) At $x = 9$ we want a circle around the x -axis of radius 3. Write parametric functions $y(\theta)$ and $z(\theta)$ for this circle.
 - (d) Write the parametric equations $x(x, \theta)$, $y(x, \theta)$ and $z(x, \theta)$ for this surface.
37. [This problem depends on Problem 36.]
- (a) Write parametric equations for the surface that results when the curve $z = \sin x$ on the xz plane is rotated about the x -axis.
 - (b)  Use a computer to graph your equations from Part (a) and confirm that the resulting surface looks right.
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38. **Finding Gravitational Force.** A uniform square of mass M extends from $(-L, -L)$ to (L, L) . A point mass m is at position $(0, 2L)$.
- Draw the square and the point mass. Include in the square a differential box. Label the position (x, y) of the box and the distance r from the box to the mass m .
 - Find the mass dm of the tiny box and the distance r from the box to the mass.
 - Find the magnitude of the gravitational force dF exerted by the tiny box on the mass m .
 - Explain how we can know that the total gravitational force of the square on the mass will be in the $-y$ direction.
 - Calculate the y component of the gravitational force dF that you found.
 - Set up a double integral over the square to add up the contributions from all the differential boxes.
 -  Evaluate the double integral to find the total gravitational force \vec{F} exerted by the square on the mass m . Express your answer in unit vector notation and check that it has correct units.

In Problems 39–43 find the gravitational force exerted by a uniform object of mass M with the specified shape on a point mass m at the specified location. Remember to use symmetry so you don't calculate more components of \vec{F} than you need to. You can look at Problem 38 as a model.

-  A thin rod stretching from $(-L, 0)$ to $(L, 0)$ on a mass at point $(0, H)$.
-  A square in the xy plane with corners at $(-L, -L, 0)$ and $(L, L, 0)$ on a mass at the point $(0, 0, H)$.
-  A disk of radius $R = 0.5$ m and mass $M = 5$ kg on a 2 kg mass a distance R from the edge of the disk (but in the same plane).
-  A 5000 kg cube of edge length $L = 10$ m on a 1 kg mass attached to one of its corners.
-  A right circular cylinder of height H and radius R on a mass along its central axis at a distance L above the bottom of the cylinder. (Assume $L > H$ so the mass m is outside the cylinder.)


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44. A thin rod of uniformly distributed mass M stretches from $(0, 0)$ to $(0, L)$. You are going to find its gravitational force on a mass m at a point (x, y) where x and y are both larger than L . (*Hint:* Be careful not to name a variable of integration y , since y already means one of the coordinates of the mass m in this problem.)
- Write integrals that represents the components of the force.
 - Calculate the y -component of the force.
 -  Calculate the x -component of the force.
 -  Calculate the magnitude of the force and take its limit as $L \rightarrow 0$. Explain why your answer makes sense.

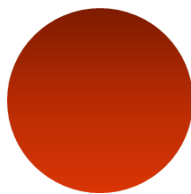
45. Exploration: Two Extended Objects

A uniform cube mass M_1 has two opposite corners at $(0, 0, 0)$ and (L, L, L) . Another uniform cube of mass M_2 has opposite corners at $(0, 0, 2L)$ and $(L, L, 3L)$. In this problem you will find the gravitational force of cube 1 on cube 2.




- Draw the two cubes and draw a tiny box inside each one. Remember that you can't use the same label for two different variables, so you should call the position of one box (x, y, z) and the other one (x', y', z') .
- Use Newton's law for the gravitational force between point particles to find the force exerted by the tiny box in cube 1 on the tiny box in cube 2.


By symmetry, the total force of cube 1 on cube 2 must be in the z direction, so we will henceforth ignore the x and y components.

- Find the z component of the force you found in Part (b).
 - Set up, but do not yet evaluate, an integral for the z component of the force of all of cube 1 on all of cube 2.
 -  That sixth-order integral is hard to evaluate, even for a computer. As it stands now you can't evaluate it numerically because the limits have L in them. Set $L = 1$ and evaluate the integral numerically. The result for F_z should still include G , M_1 , and M_2 . Using that result and the requirement that the answer have units of force, find F_z in terms of G , M_1 , M_2 , and L .
46. Uh-oh, my printer is running out of ink! I tried to print a solid red circle, but the further the paper went in, the less ink I had, and the circle came out looking like this.



Glancing at it I notice that the ink is very thick on top, and gradually paler as the circle progresses. Measuring carefully I discover that the circle has a 5 inch radius and that the density of ink is given by approximately 2^h dots per square inch, where h is the height above the center of the circle in inches. (So at the top of the circle, $h = 5$; at the bottom, $h = -5$.)

- Calculate the density of dots at the very bottom of the circle. If the entire circle had that density, how many dots would it have?
 - Draw a differential slice of the circle. Of course, your slice must have more or less a uniform density!
 - Compute the area of your slice, as a function of h and dh only.
 - Compute the number of dots in your slice.
 - Set up an integral to represent the number of dots in the entire circle.
 -  Evaluate the integral. Give your answer rounded to the nearest integer.
47.  Evaluate the integral of $r \cos^2 \theta$ (where r and θ are the usual spherical coordinates) in the region bounded by the xy -plane and the cone $(z - 1)^2 = x^2 + y^2$.
48.  Evaluate the integral of r (distance from the origin) over a cylinder of radius R and height H with its base centered on the origin.

49.  A cylinder of radius 2 m and height 1.5 m is filled with a gas. The density of the gas drops exponentially with distance from the central axis, going from 2.5 kg/m^3 on the central axis to $1/5$ that density at the outer edge.
- (a) Write the density of the gas in Cartesian coordinates. If you use any letters (other than the coordinates) you should specify what their numerical values are.
 - (b) Set up a Cartesian integral for the moment of inertia of the cylinder about its central axis.
 - (c) Evaluate the integral numerically to find the moment of inertia of the gas.
 - (d) Repeat Parts (a)–(c) in cylindrical coordinates. You should be able to verify that the integral appears different in the two coordinate systems, but gives the same answer.