A Final Section for Linear Algebra
Putting It Together: Revisiting the Three-Spring Problem

The linear algebra motivating exercise walked through the problem of two balls connected by three springs, but left a lot of gaps in the math. In this section we circle back to that problem with the tools of linear algebra now firmly in hand to fill in the gaps.

Explanation: Revisiting the Three-Spring Problem

In this explanation we are going to solve two "coupled differential equations" problems. The first solution will closely parallel the process we used in the linear algebra motivating exercise; it makes use of matrices and eigenvalues, but fundamentally works in the basis of the two positions x_1 and x_2. At the very end it uses the normal-mode basis to find the solution for a particular set of initial conditions. The second process translates the entire problem into the basis defined by the eigenvectors—the normal modes—right from the start. In this natural basis we find that the problem almost solves itself.

In both cases we have changed the original differential equations so the answers will come out different. Following both solutions carefully is a great way to make sure you understand the linear algebra you've learned in this unit.

The first approach, illustrated in the boxed example below, uses matrices in three key steps.

1. We begin by rewriting the problem as a matrix equation. Two characteristics of this problem suggest such an approach: the state of the system requires multiple variables, and the derivatives are based on linear combinations of those variables.
2. We then find the normal modes. Each normal mode is characterized by a particular ratio of x_2 to x_1 and a particular frequency of oscillation; the former is given by an eigenvector, and the latter by its associated eigenvalue. We are not emphasizing here "how to find eigenvectors and eigenvalues" (although that's a good thing to know). Focus instead on how the definition of an eigenvector transforms the coupled equations into two decoupled differential equations that we can solve individually.
3. Finally we combine those normal modes, which amounts to a change of basis. A matrix converts "this much of the first normal mode and that much of the second" to "this is x_1 and this is x_2," and its inverse matrix converts the other way to find the particular solution for given initial conditions.

Example: Eigenvectors and Coupled Equations

Question: Find the solution to the equations

$$\ddot{x}_1 = -3x_1 - 2x_2$$
$$\ddot{x}_2 = -x_1 - 2x_2$$

with initial conditions $x_1(0) = 3$, $x_2(0) = -9$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

Answer: We begin by rewriting the problem as a matrix equation.

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

The next step is to find the two eigenvectors of the 2×2 matrix in Equation 1. The first, which you can find by hand or on a computer, is $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ with eigenvalue -4. Any multiple of an eigenvector is also an eigenvector, so that first eigenvector tells us that if you plug in any state in which $x_2 = (1/2)x_1$, the matrix will simply multiply that state by -4. So if we replace x_2 with $(1/2)x_1$ the equation becomes:

$$\begin{pmatrix} \ddot{x}_1 \\ (1/2)\ddot{x}_1 \end{pmatrix} = -4 \begin{pmatrix} x_1 \\ (1/2)x_1 \end{pmatrix}$$

So we now have the equation $\ddot{x}_1 = -4x_1$. (In fact we have this same equation twice.) The solution to this equation becomes the first normal mode.

$$x_1(t) = A_1 \cos(2t) + A_2 \sin(2t), \quad x_2(t) = \frac{1}{2} x_1(t) \tag{2}$$

The other eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue -1, which becomes the other normal mode.

$$x_1(t) = B_1 \cos t + B_2 \sin t, \quad x_2(t) = -x_1(t) \tag{3}$$

The general solution to our initial problem is a sum of these two normal modes. Next we apply our initial conditions. Knowing that the initial velocities are zero we discard the sine terms. We can easily write a matrix that converts normal mode amplitudes to initial positions; inverting that matrix allows us to find the normal mode amplitudes that match our given initial positions. As with the eigenvectors above, we're not showing you here the process of finding an inverse matrix; we're showing you what to do with it.

$$\text{The inverse of} \quad \begin{matrix} & A_1 & B_1 \\ x_1(0) & 1 & 1 \\ x_2(0) & 1/2 & -1 \end{matrix} \quad \text{is} \quad \begin{matrix} & x_1(0) & x_2(0) \\ A_1 & 2/3 & 2/3 \\ B_1 & 1/3 & -2/3 \end{matrix}$$

Applying this matrix to the initial conditions $x_1(0) = 3$, $x_2(0) = -9$ gives $A_1 = -4$, $B_1 = 7$, so the solution is

$$x_1(t) = -4\cos(2t) + 7\cos t$$
$$x_2(t) = -2\cos(2t) - 7\cos t$$

If there are any steps in that solution that you didn't follow, we hope you can find the appropriate part of your book or notes to go back and review. Pay particular attention to the way each eigenvector of the matrix became a normal mode of the system; because the matrix multiplication became a scalar multiplication, the solution was two different oscillations with the same frequency.

The process we just modeled is all you need to solve this type of coupled oscillator problem. In the rest of this section we are going to show you a slightly different way of writing the steps in that solution. This second approach illustrates the meaning of basis states and diagonalization more directly than the method we just used.

Above we approached Equation 1 by plugging in the eigenvectors one at a time, and seeing what they did to the equation. Instead, let's rewrite the entire equation in the basis defined by the eigenvectors. So the state of the system—the positions of the balls, which we have previously expressed as x_1 and x_2—we will now express like this.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = f(t) \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + g(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4}$$

You can interpret Equation 4 physically. As we have said all along, instead of asking "where is each ball?" we ask "how much of each normal mode do we have?" But right now let's focus on two mathematical questions.

Can we do that? Yes we can, for absolutely any functions x_1 and x_2, and here's why. At any given time t, the values x_1 and x_2 represent a vector. Equation 4 tells us that we can express that vector as a linear combination of the two eigenvectors. We know we can do that, *not* because they are eigenvectors, but just because they are two non-parallel vectors and therefore form a basis. The numbers f and g represent the coefficients in this new basis. At a later time t the values x_1 and x_2 are different; we are converting them to the *same* basis with *different* coefficients. That is why f and g are functions of t.

What happens when we do that? When we express all our states in a different basis, the entire equation—including the transformation matrix itself—changes. When you use the eigenvectors as a basis, the transformation is represented by a diagonal matrix of the eigenvalues. (This fact comes very directly from the definition of an eigenvector.) And a diagonal matrix always *decouples the equations.* Equation 1 becomes:

$$\begin{pmatrix} \ddot{f} \\ \ddot{g} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{5}$$

Our hope is that you understand why this equation must be true given that $\begin{pmatrix} f \\ g \end{pmatrix}$ is the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ expressed in the natural basis of the transformation matrix, but you can derive it more directly by plugging Equation 4 into Equation 1. See Problem 16.

This gives us the two equations $\ddot{f} = -4f$ and $\ddot{g} = -g$. Solving those, and plugging the solutions into Equation 4, leads us back to the solutions we found before.

When we used this approach to solve the three-springs problem the normal modes ended up being sine waves, but once you reduce the problem to a set of decoupled ODEs you can solve them to find the normal modes no matter what kind of function they are. Also, instead of solving for a particular set of initial conditions as we have you can always choose to add the normal modes with their arbitrary constants to get the general solution. Both of these points are illustrated in the example below.

Example: Diagonalization and Coupled Equations

Question: Find the general solution to the equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

(Notice that these are first derivatives, so this is different from our previous example, but the same techniques apply.)

Answer: The eigenvectors and eigenvalues of this transformation matrix are $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ with corresponding eigenvalues -5 and 2 respectively. We can write x and y in this basis as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f(t) \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + g(t) \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{7}$$

Equation 7 says "you can express vector (x, y) at any given time as a linear combination of these two basis vectors, with f and g as the coefficients." You can see what this conversion does to Equation 6 by directly plugging in these expressions for $x(t)$ and $y(t)$ (Problem 17) or by converting the matrix to this new basis, but you don't have to: we already know that in the basis defined by the eigenvectors, a transformation becomes a diagonal matrix of the eigenvalues. (Such a matrix says "multiply the $\vec{V}_1$ component by λ_1" and so on.) So Equation 6 becomes:

$$\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

The "natural basis" for this transformation has done its job, giving us the *decoupled* differential equations $\dot{f} = -5f$ and $\dot{g} = 2g$ with solutions $f = Ae^{-5t}$ and $g = Be^{2t}$. Finally, we can go back to the xy representation to write the general solution.

$$x = Ae^{-5t} + Be^{2t}, \quad y = \frac{1}{2}Ae^{-5t} - 3Be^{2t}$$

Stepping Back

As you may have guessed, we want you to be able to solve the three-spring problem. Coupled oscillators come up a lot. If you boil the whole thing down to "the resonant frequencies of coupled oscillators are the square roots of the absolute values of the eigenvalues," you will have a tidbit that may prove useful.

But beyond that, of course, our whole strategy in the motivating exercise and here has been to use the three-spring problem as both introduction and capstone to your understanding of linear algebra. It is for that reason that we just solved the same problem twice. In both cases, the takeaway is "the eigenvectors of the matrix define the normal modes of the system."

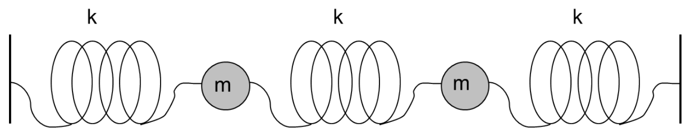
In the first approach, this fact came very directly from the eigenvalue equation. When you replace the state of the system with an eigenvector, the two variables get separate differential equations but with the same frequency, which defines a normal mode. If you try replacing the state of the system with something that isn't a normal mode (say, $x_2 = 3x_1$) you will end up with two contradictory differential equations.

In the second approach, we rewrote the entire problem in the basis defined by the normal modes. Instead of solving directly for the positions of the balls, we solved for the amplitudes of the normal modes. This is less direct, but it gives you perhaps the clearest view of why these normal modes work and no others would; only in the basis of the normal modes does the transformation matrix become diagonal, thus decoupling the two equations.

If you find the second approach confusing start by getting comfortable with the first approach. You should be able to understand that and solve problems with it, but once you have worked with it a while come back to the second approach and see if you can see how it is doing the same thing as the first one, but in a different basis.

Problems: Revisiting the Three-Spring Problem

1. The figure below shows a set of coupled oscillators. For this problem take $k = 8$ N/m, $m = 2$ kg.



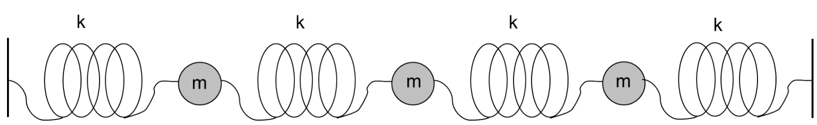
In later problems you'll solve for the normal modes of systems like this, but for this one we're just going to tell you that one of them is represented by the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue 2, and the other is represented by the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue $2\sqrt{3}$.

   (a) In the first normal mode both balls move to the right and left in unison, always maintaining the same displacement from their equilibrium points. Explain why it makes sense physically that the two balls could move together in this way. *Hint*: think about the forces on the two balls, and in particular think about whether the middle spring is stretched, compressed, or neither in this normal mode.

   (b) In Part (a) we described what one normal mode physically represents. Give a similar description of the other normal mode. (You don't have to do what you did in that part; just do the part we did.)

   (c) Explain why the normal mode in Part (b) has a higher frequency than the one in Part (a). Your answer shouldn't be about eigenvalues or normal mode frequencies, but about springs and forces.

2. The image below shows three coupled oscillators.



You should be able to write down their equations of motion and solve them to find the normal modes, but in this problem we want to focus on the physical interpretation so we're going to give you the eigenvectors of the transformation matrix.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

(These come out the same regardless of the values of k and m.) The first of these eigenvectors describes a normal mode in which you pull all three balls to the right (or left), pulling the middle one slightly farther than the outer two, and then let go. They will then oscillate that way forever, always moving to the right and left together, with the middle one oscillating with a larger amplitude than the other two. Using that description as a guide, write similar descriptions for what the other two eigenvectors physically represent.

3. **Walk-Through: Coupled Differential Equations, First Method.** In this problem you will find the solution to the equations

$$\ddot{x} = -6x + y$$
$$\ddot{y} = -4x - y$$

using the approach outlined in the example on Page 2.

   (a) Rewrite these differential equations as a single matrix equation. One side will be the column matrix $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$. The other side will be a square matrix of all numbers times a column matrix that represents the state of the system.

(b) Find the eigenvectors and eigenvalues of the square matrix in your equation. You may do this with a computer or by hand.

(c) Choose one of the eigenvectors you found and use it to write y as a function of x. Plug this function into your matrix equation from Part (a) and multiply the matrices. The result should give you two equivalent differential equations for x.

(d) Solve your equation for x. The solution should contain two arbitrary constants. Then use the relationship between x and y in this eigenvector to write the solution for y, which will contain the same two constants.

(e) Repeat Parts (c)–(d) for the other eigenvector.

(f) Combine your answers to find the general solution for x and y.

(g) Plug your general solution in to the original differential equations and verify that it works.

4. *[This problem depends on Problem 3.]*

(a) Assume that $\dot{x}(0) = \dot{y}(0) = 0$. Use this constraint to simplify your solution. (Several terms should drop out.) Assume these initial velocities are zero throughout the rest of this problem.

(b) Write a set of initial conditions for x and y for which the system will be in one of its normal modes. Write the solution $x(t)$, $y(t)$ corresponding to those initial conditions. This should require no calculations.

(c) Suppose your system is in a mix of the first normal mode with amplitude A plus the second normal mode with amplitude B. Write the matrix that converts the vector $\begin{pmatrix} A \\ B \end{pmatrix}$ to the initial conditions $x(0)$ and $y(0)$.

(d) Using the matrix you found in Part (c), write the initial conditions that would give you $A = B = 2$.

(e) Write the matrix that converts from the initial conditions $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ to the normal mode amplitudes A and B.

(f) The system starts with $x(0) = 3$, $y(0) = 4$. Use the matrix you wrote in Part (e) to find A and B and use that to write the solution $x(t)$, $y(t)$ for these initial conditions.

5. **Walk-Through: Coupled Differential Equations, Second Method.** In this problem you will find the solution to the equations

$$\ddot{x} = -3x - 4y$$
$$\ddot{y} = -3x + y$$

using the approach illustrated in the example on Page 4.

(a) Rewrite these differential equations as a single matrix equation. One side will be the column matrix $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$. The other side will be a square matrix of all numbers times a column matrix that represents the state of the system.

(b) Find the eigenvectors and eigenvalues of the square matrix in your equation. You may do this with a computer or by hand.

(c) Write the vector $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ as a sum of the eigenvectors you wrote in Part (b). The coefficients in this sum will be functions of time, which you can call $f(t)$ and $g(t)$.

(d) The equation you wrote in Part (a) should have had a column matrix on the left and a square matrix and a column matrix on the right. All of these matrices were written in the xy basis. Rewrite the equation in the fg basis. The result should be two decoupled differential equations.

(e) Solve the equations to find $f(t)$ and $g(t)$. Include all arbitrary constants.

(f) Use your solutions for f and g and your answer to Part (c) to write the general solution $x(t)$, $y(t)$.

6. Solve the differential equations in Problem 3 using the approach illustrated in the example on Page 4.

7. Solve the differential equations in Problem 5 using the approach outlined in the example on Page 2.

In Problems 8–13 find the normal mode solutions for the given sets of equations. Then find the amplitude of each normal mode if the system starts with the given initial conditions. When writing the solutions you should include all arbitrary constants, but when you plug in initial conditions for second order equations assume the initial first derivatives are zero. In all cases you may use a computer to find eigenvectors and eigenvalues or do it by hand, but in the case of larger matrices we've marked the problems with a computer icon to indicate that doing it by hand would be tedious.

8. $\ddot{x} = -7x + 2y$, $\ddot{y} = -4x - y$, $x(0) = 1$, $y(0) = 3$
9. $\ddot{x} = -3x - y$, $\ddot{y} = -4x - 3y$, $x(0) = 1$, $y(0) = -5$
10. $\ddot{x} = -4x + y$, $\ddot{y} = x - 4y$, $x(0) = 1$, $y(0) = -1$
11. $\dot{x} = 2x + y$, $\dot{y} = 4x + 5y$, $x(0) = 2$, $y(0) = 1$. *Notice that these are first derivatives.*
12. $\ddot{x} = 3x + y + z$, $\ddot{y} = x + y + 3z$, $\ddot{z} = -x + 3y + z$, $x(0) = 1$, $y(0) = 2$, $z(0) = 3$ *Hint:* For positive eigenvalues you can solve the ODEs using exponentials, but it's easier to apply the condition that the initial derivatives are zero if you write the solution in terms of sinh and cosh. If you have no idea what those are just go ahead and use exponentials and it won't be too bad.
13. $\dot{x} = -2x - y + z$, $\dot{y} = -2x + y - 3z$, $\dot{z} = x + y - 2z$, $x(0) = 1$, $y(0) = 1$, $z(0) = 1$. *Notice that these are first derivatives.*

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14. In the boxed example on Page 2 we solved a coupled oscillator problem on the assumption that the balls started with zero velocity. In this problem you will relax that assumption. Your starting point will be Equations 2–3.
    (a) The four initial conditions are $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$. If $A_1 = 1$ and the other three arbitrary constants are all zero, what are the values of these four initial conditions? Answer the same question for each of the other three arbitrary constants.
    (b) Using your answers to Part (a), write a 4×4 matrix to convert from the four arbitrary constants to the four initial conditions.
    (c) Invert that matrix to find a matrix that converts from initial conditions to arbitrary constants. (If you think about this there's a relatively simple way to do it by hand, but you're welcome to use a computer if you prefer.)
    (d) Write down the solution for the initial conditions $x_1(0) = 3$, $x_2(0) = 1$, $\dot{x}_1(0) = 1$, $\dot{x}_2(0) = -2$.
15. In the boxed example on Page 2 our solution required finding the eigenvectors and eigenvalues of $\begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}$. Find those eigenvectors and eigenvalues without a computer.
16. In the explanation we said that when you rewrote x_1 and x_2 as a sum of the eigenvectors of the transformation matrix the matrix would become diagonal. That must be true by the definition of eigenvectors, but Equation 5 may still look a bit mysterious. In this problem you'll derive that equation more rigorously.
    (a) Rewrite the right hand side of Equation 4 as a square matrix times a column matrix. This should only require looking at the right hand side and remembering what matrix times column means.
    (b) Plug Equation 4, with the right-hand side rewritten as a matrix times column, into Equation 1. The result should be a matrix equation with a square matrix times a column matrix on the left, and a square matrix times a square matrix times a column matrix on the right.
    (c) Multiply the two square matrices on the right, so that your equation is now a square matrix times a column matrix on each side.

(d) Multiply both sides of the equation by the inverse of the square matrix on the left of the equation, and carry out the multiplication with the square matrix on both sides. The resulting equation should be Equation 5.

17. In the boxed example on Page 4 we converted coupled differential equations for x and y into decoupled differential equations for f and g based on the properties of eigenvectors. For the moment, let's pretend you had never heard of an eigenvector. Starting with the equations $\dot{x} = -4x - 2y$ and $\dot{y} = -3x + y$, make the substitutions $x = f + g$ and $y = (1/2)f - 3g$ and see what you get. (We hope it will look like what we got!)

18. The techniques in this section can be applied to a variety of differential equations that don't look like the ones we've solved so far. As an example, you'll use them in this problem to solve the single damped oscillator equation $\ddot{x} + 4\dot{x} + 3x = 0$.

(a) Define a variable $v(t) = \dot{x}(t)$. Rewrite the damped oscillator equation as an equation for $\dot{v}$ in terms of x and v.

(b) The equation you just wrote, along with the equation $\dot{x} = v$, are a pair of linear, coupled differential equations. Use the methods of this section to find the general solution $x(t)$.

19. In this problem you will solve the equations

$$\dot{x} = -9x - 10y$$
$$\ddot{y} = -7x - 6y$$

(a) Define a variable $v(t) = \dot{y}(t)$. Rewrite the equation for $\ddot{y}$ as an equation for $\dot{v}$.

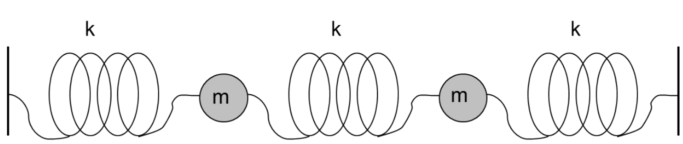
(b)  Including the equation $\dot{y} = v$ you should now have three coupled first-order differential equations for x, y, and v. Use the methods of this section to find the general solution to these equations. (The only part you should need the computer for is finding the eigenvectors and eigenvalues of a 3×3 matrix.)

20. Explain why you cannot use the techniques of this section to solve the following differential equations:

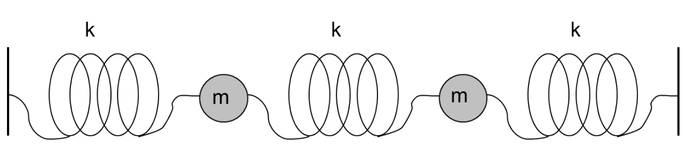
$$\ddot{x} = -x^2 - 3y$$
$$\ddot{y} = x + 2y$$

In Problems 21–26, write the equations of motion for the system and find the normal modes. In each case you'll need to start by writing expressions for the force on each ball from each of the two springs touching it. That force will depend on how stretched or compressed the spring is, which in turn will depend on the positions of the two balls touching the spring (or just one if the spring is connected to the wall). As an example, consider the three-spring problem we solved in the explanation. The leftmost spring is stretched by x_1, so it exerts a force $-k_1 x_1$ on Ball 1. The middle spring is stretched by $x_2 - x_1$, but if it is stretched it pulls Ball 1 to the right, so its force on Ball 1 is $k_2(x_2 - x_1)$. Putting those together and dividing force by mass to get acceleration gives $\ddot{x}_1 = -(k_1/m_1)x_1 + (k_2/m_1)(x_2 - x_1)$. If you plug in the numbers given in the explanation this will give you the differential equation we wrote for x_1. You can similarly derive the equation for x_2.

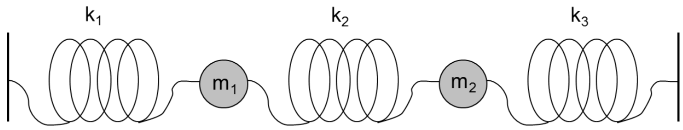
21. $k = 10$ N/m, $m = 3$ kg



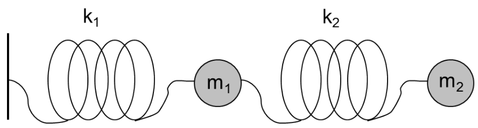
22. Leave k and m as letters in your solution.



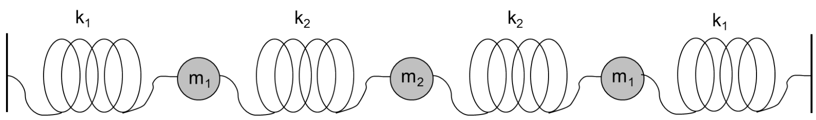
23. $k_1 = 16$ N/m, $k_2 = 4$ N/m, $k_3 = 10$ N/m, $m_1 = 2$ kg, $m_2 = 2$ kg



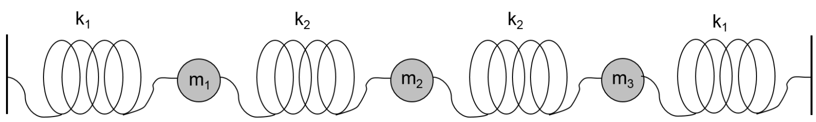
24. $k_1 = 3$ N/m, $k_2 = 2$ N/m, $m_1 = 1$ kg, $m_2 = 1$ kg



25.  $k_1 = 24$ N/m, $k_2 = 6$ N/m, $m_1 = 3$ kg, $m_2 = 2$ kg. *The only thing you should need a computer for is the eigenvectors and eigenvalues of a* 3×3 *matrix.*



26. $k_1 = 24$ N/m, $k_2 = 6$ N/m, $m_1 = 3$ kg, $m_2 = 1$ kg, $m_3 = 6$ kg



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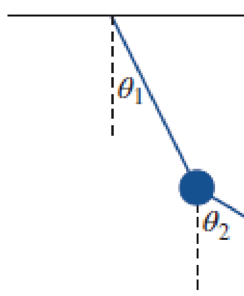
27. A double pendulum is shown below.



The equations describing this system depend on the angles θ_1 and θ_2, the gravitational acceleration g, and the length L of the strings.

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + 2\frac{g}{L} \sin\theta_1 = 0$$
$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{g}{L} \sin\theta_2 = 0$$

These equations are nonlinear and cannot be solved using the methods in this section. For small oscillations, however, you can assume θ_1, θ_2, and all of their derivatives remain small, and thus approximate this with a set of linear equations.

(a) Replace all the trig functions with the linear terms of their Maclaurin series expansions. Then eliminate any remaining nonlinear terms. The result should be two coupled, linear differential equations.

(b) Solve the equations algebraically for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ so you can write them in the form we used in this section.

(c) Find the normal mode frequencies of this system for small oscillations.

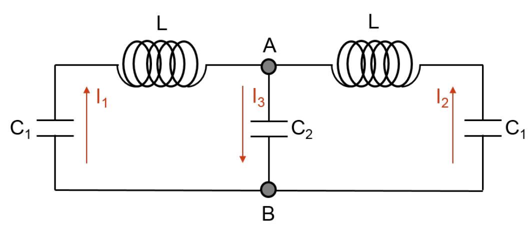
28. *[This problem depends on Problem 27.]* For this problem take $g = 9.8$ m/s and $L = 1.0$ m and assume in all cases that the two pendulums start at rest.

(a) Choose one of the two normal modes of the system and numerically solve the original equations (not the linearized ones) for $\theta_1(0) = 0.1$, with $\theta_2(0)$ chosen to be whatever it needs to be for that normal mode. Plot $\theta_1(t)$ and $\theta_2(t)$ together on one plot. Include enough time on your plot to see at least 3 full oscillations of the pendulums.

(b) Repeat Part (a) for values of $\theta_1(0) = 0.2$, 0.3, and so on up to 1. In each case adjust $\theta_2(0)$ to whatever the normal mode solution says it should be.

(c) Describe what the system's behavior looks like for small and large amplitudes using these initial conditions. Explain why the different behavior in the different cases makes sense.

29. The figure below shows a circuit with capacitors and inductors.



Conservation of charge requires that $I_3 = I_1 + I_2$, so there are only two variables needed to describe the current in the system. You can get equations for those currents by setting the voltage drop from point A to point B equal along all three paths and differentiating: $I_1/C_1 + L\ddot{I}_1 = I_2/C_1 + L\ddot{I}_2 = -(I_1 + I_2)/C_2$. (Don't worry if you didn't follow how we got that; your job is going to be to solve it.) Remember that $\ddot{I}$ means d^2I/dt^2.

(a) Write these as a pair of equations for $\ddot{I}_1$ and $\ddot{I}_2$.

(b) Find the normal modes of the system and use them to write the general solution.

(c) In one of the normal modes the current on both sides of the circuit is equal, so at some point in time a current I is flowing up on the left, an equal current I is flowing up on the right, and a current $2I$ is flowing down the middle. Then they reverse—down on both sides and up in the middle—and go back and forth like that forever. Using this description as a guide, describe the physical state represented by the other normal mode.