Motivating Exercise for Linear Algebra:
All I Really Need to Know About Matrices I Learned from the Three-Spring Problem

The figure shows two balls connected to each other and to the walls by three springs. We will assume throughout this exercise that the balls begin at rest, so our initial conditions are only their starting positions $x_1(0)$ and $x_2(0)$. As an example the drawing shows Ball 1 at its equilibrium position and Ball 2 displaced to the right.

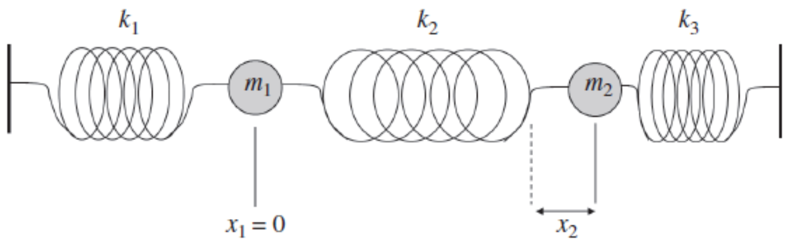


Figure 1: The three-spring problem with Ball 1 at equilibrium and Ball 2 displaced to the right.

With a bit of introductory physics (Problem 10) you can derive the appropriate differential equations. For $m_1 = 2$ kg, $m_2 = 9$ kg, $k_1 = 2$ N/m, $k_2 = 6$ N/m, and $k_3 = 21$ N/m, the equations become:

$$\frac{d^2x_1}{dt^2} = -4x_1 + 3x_2 \tag{1}$$

$$\frac{d^2x_2}{dt^2} = \frac{2}{3}x_1 - 3x_2 \tag{2}$$

We begin by noticing that Equation 1 writes $\ddot{x}_1$ as a *linear combination* of x_1 and x_2, and Equation 2 does the same for $\ddot{x}_2$. (If you're not familiar with that notation, $\dot{x}$ means dx/dt and $\ddot{x}$ means d^2x/dt^2 and so on.) In general, a linear combination of x and y is any function $ax+by$ where a and b are constants. Linear algebra is all about working with that particular kind of function, using a mathematical tool called a "matrix."

When you learn how to rewrite linear equations with matrices, you will find that you can rewrite Equations 1 and 2 as a single matrix equation. This simple change in notation will allow you to bring all the tools of matrix mechanics to bear on this problem.

Throughout this exercise you will see other arrows like the one above. Each one discusses a topic where you will learn how to use matrices to solve part of this problem. In some cases your first reaction may be "I don't need matrices to solve this." But this example has only two equations with two unknowns: engineers frequently work with hundreds of equations and unknowns, and matrices give us the approach that scales up to that size.

Two simple solutions for the just-right initial conditions

For any value of A the following functions provide a valid solution to Equations 1 and 2.

$$\begin{aligned} x_1(t) &= A\cos\left(\sqrt{2}\ t\right) \\ x_2(t) &= \frac{2}{3}A\cos\left(\sqrt{2}\ t\right) \end{aligned} \tag{3}$$

You can easily verify that Equation 3 works by plugging it into Equations 1–2, but this isn't just any solution: it's a particularly important type of solution called a "normal mode." The whole strategy of solving problems like this is based on knowing how to find the normal modes, and how to use them once you find them. We will provide sketchy outlines of both those processes in this exercise, which will be fleshed out as you learn about linear algebra. But first it is essential for you to understand the kind of behavior that Equations 3 describe.

Let's start by considering the initial conditions. Equation 3 implies four of them: $x_1(0) = A$, $x_2(0) = (2/3)A$, and both $\dot{x}_1(0)$ and $\dot{x}_2(0)$ are zero.[1] This tells us how to get the system into this mode. You pull m_1 out to any distance you like, right or left. Then you pull m_2 out to exactly 2/3 of that distance in the same direction. Then you release both masses from rest.

And what will happen from there? The two masses will oscillate with different amplitudes but the same frequency. They will go through $x = 0$ (their equilibrium positions) at the same time, reach their maximum amplitudes on the other side at the same time, and come back to their starting positions at the same time. *Because they oscillate with the same frequency and phase, they will maintain the relationship $x_2 = (2/3)x_1$ at all times.*

So this normal mode is defined by two quantities: the ratio of x_2 to x_1, and the frequency with which the system oscillates. If the two positions are in that ratio, they will stay in that ratio and oscillate with that frequency forever. All this behavior is based on the fact that the two masses are oscillating with the same frequency as each other.

Definition: Normal Modes

A normal mode of this system is a solution in which the ratio x_2/x_1 stays constant over time. This implies that the two balls are oscillating with the same frequency, so the motion of the entire system is periodic.

As we will see below, most possible behaviors of this system do not qualify as normal modes. For most initial conditions the ratio x_2/x_1 will not remain constant and the balls will not oscillate with the same frequency (or with any constant frequency). But there is one other normal mode, described by the following solution.

$$\begin{aligned} x_1(t) &= B\cos\left(\sqrt{5}\,t\right) \\ x_2(t) &= -\frac{1}{3}B\cos\left(\sqrt{5}\,t\right) \end{aligned} \tag{4}$$

Once again this solution represents a very specific set of initial conditions: you pull Ball 1 a certain distance from equilibrium, pull Ball 2 a third of that distance in the *opposite* direction, and release them from rest. The balls move inward at the same time, and then outward. But once again the frequencies are the same, the motion is simple and periodic, and the ratio $x_2 = -(1/3)x_1$ persists.

You might now expect that for every initial x_2/x_1 there is a certain normal mode with a certain frequency associated with it, but in fact Equations 3 and 4 represent the only two normal modes of this particular system. There are no more.

So we've learned a lot about our system, but you're probably not ready to sound the victory bells. First of all, what if x_2 doesn't happen to start at $(2/3)x_1$ or $-(1/3)x_1$? And second of all, how did we find those solutions in the first place? Below we will outline the answers to both of those questions, but we will have to leave a number of holes that will be filled in with matrices.

More complicated solutions for other initial conditions

Let's see how our system responds to a variety of initial conditions.

[1] If the initial velocities were nonzero we would need to add sine solutions as well as cosines: see Problem 12.

First Example: $x_1(0) = 12$ **and** $x_2(0) = -4$

This fits our second normal mode, where $B = 12$.

$$x_1(t) = 12\cos\left(\sqrt{5}\ t\right)$$
$$x_2(t) = -4\cos\left(\sqrt{5}\ t\right)$$

The balls will oscillate with a frequency of $\sqrt{5}$ and with amplitudes 12 and 4. Their displacements will always be in opposite directions (out of phase by π).

Second Example: $x_1(0) = 4$ **and** $x_2(0) = 5/3$

There is no way to get *either* of our two solutions to fit this initial condition, and there is no normal mode with $x_2/x_1 = 5/12$. It seems that our solutions so far are only useful in a few very carefully selected cases.

But Equations 1 and 2 are linear, homogeneous differential equations, so any combination of solutions is a solution. If we choose $A = 3$ in the first normal mode and $B = 1$ in the second normal mode, and then *add the solutions*, we get a new solution.

$$\begin{aligned} x_1(t) &= 3\cos\left(\sqrt{2}\ t\right) + \cos\left(\sqrt{5}\ t\right) \\ x_2(t) &= 2\cos\left(\sqrt{2}\ t\right) - \frac{1}{3}\cos\left(\sqrt{5}\ t\right) \end{aligned} \tag{5}$$

In Problem 13 you will plug these in to verify that they solve the differential equations and our initial condition. But it's more important to see that they work *without* plugging them in, by following this logic.

- The solution $x_1 = 3\cos(\sqrt{2}\ t)$, $x_2 = 2\cos(\sqrt{2}\ t)$ solves the differential equations; we know this because it fits our first normal mode. It meets the initial conditions $x_1(0) = 3$, $x_2(0) = 2$.
- The solution $x_1 = \cos(\sqrt{5}\ t)$, $x_2 = -(1/3)\cos(\sqrt{5}\ t)$ solves the differential equations because it is our second normal mode. It meets the initial conditions $x_1(0) = 1$, $x_2(0) = -1/3$.
- When we add these two solutions the result *must* still satisfy the differential equations, and must meet initial conditions $x_1(0) = 4$, $x_2(0) = 5/3$.

The "Second Example" above is the most important part of this exercise, capturing an entire solution in the two numbers $A = 3$ and $B = 1$. In other words the system starts out in a state that can be represented as 3 of the first normal mode plus 1 of the second normal mode. We know how each of these normal modes evolves in time, so we know how the entire system will evolve. Hence, the question "how do I find the behavior of this system?" is replaced with "how do I express the initial state as a linear combination of the two normal modes?"

When you learn to multiply matrices, you will have an elegant formalism for converting from normal modes to positions. This process can also be framed as a "change of basis," and you will learn the conditions under which such a conversion can reliably be done.

We need to make one more point about the above solution: the resulting motion is not simple. If you were to watch the two balls the motion would look almost random, and it would not be periodic no matter how long you looked. But hidden under the randomness would be two simple, periodic systems being added together.

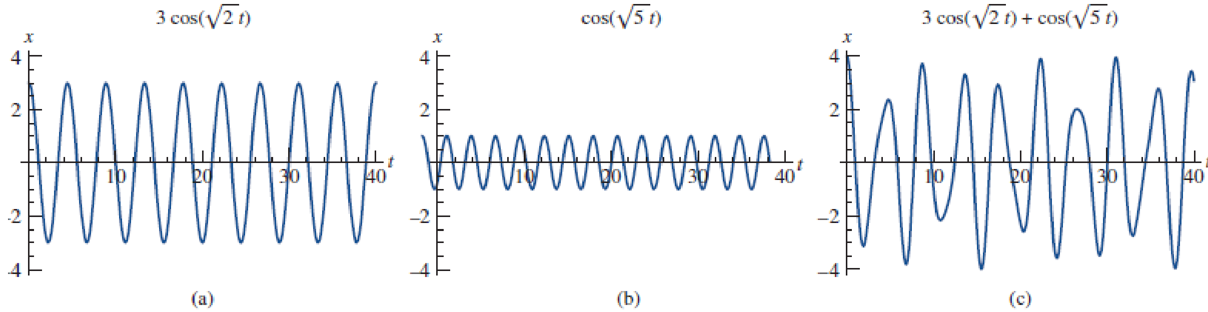


Figure 2: The two plots on the left show normal modes, which are periodic. The plot on the right shows the motion of $x_1(t)$ in the second example above. It's a sum of periodic motions, but it is not periodic.

The moral of Figure 2 is that if you look at the system and ask "Where is Ball 1 and where is Ball 2 over time?" the answer looks chaotic. You see the underlying order if you ask instead "How much of the first normal mode and how much of the second normal mode do I have?"

Third example: $x_1(0) = 5$ **and** $x_2(0) = 7$

We chose those numbers precisely because there doesn't seem to be any obvious way to get there from our two normal modes. But once again, you can do so if you choose the right values of A and B and then add solutions. In this case, the right values are $A = 26/3$ and $B = -11/3$.

$$x_1 = \frac{26}{3}\cos\left(\sqrt{2}\,t\right) - \frac{11}{3}\cos\left(\sqrt{5}\,t\right)$$
$$x_2 = \left(\frac{2}{3}\right)\frac{26}{3}\cos\left(\sqrt{2}\,t\right) - \left(-\frac{1}{3}\right)\frac{11}{3}\cos\left(\sqrt{5}\,t\right)$$

Finding the numbers 26/3 and −11/3 in the last example required us to find just the right linear combination of solutions to match the desired initial conditions. In other words, it requires us to translate from "positions of the balls" to "amplitudes of the normal modes." When you learn about inverse matrices, you will know how to find the matrix to accomplish this.

Now, how did we come up with those normal modes?

Remember that this is an outline, not a full derivation. Many of the key steps will have to be filled when you have learned more about matrices.

As we usually do with differential equations, we begin by thinking about what kind of solution we can guess. You can imagine trying any pair of functions $x_1(t)$ and $x_2(t)$, but we're going to restrict our guess in two important ways. First, given that we're describing balls on springs, we're going to guess oscillatory solutions. Second, less obviously, we're going to guess solutions for which both balls will oscillate with the same frequency: that is, normal modes. As we have seen, they are simple to understand physically and to work with mathematically, and they can combine to give us the more complicated behavior that may also arise.

Let's start by guessing a very simple normal mode: $x_1 = C_1 \cos t$ and $x_2 = C_2 \cos t$. Plugging these into Equations 1 and 2 and simplifying a little gets us here:

$$\begin{aligned} 3C_1 - 3C_2 &= 0 \\ \frac{2}{3}C_1 - 2C_2 &= 0 \end{aligned} \tag{6}$$

Matrix "row reduction" provides the right tool for solving simultaneous linear equations. (Remember again that we need to be able to do this with 100 equations, not just with two as in this example.)

You can probably solve Equations 6 in a minute or two, but we'll save you the trouble and tell you that $C_1 = C_2 = 0$. (Looks obvious now, doesn't it?) That leads us to $x_1(t) = x_2(t) = 0$, sometimes called the "trivial solution": it does satisfy Equations 1 and 2, but it isn't very helpful. We want to know how the balls move, not how they stand still.

So "$\cos t$" didn't work out too well. What if we change the frequency? In Problem 16 you will guess oscillatory functions with many different frequencies, but they will lead back to the same place.

So here comes a particularly lucky guess: what if we just happen to try $x_1 = C_1 \cos(\sqrt{2}\ t)$ and $x_2 = C_2 \cos(\sqrt{2}\ t)$? Then we end up here.

$$2C_1 - 3C_2 = 0 \tag{7}$$

$$-\frac{2}{3}C_1 + C_2 = 0 \tag{8}$$

Once again $C_1 = C_2 = 0$ fits, but this time it isn't the only solution. Equation 7 can be rewritten as $C_2 = (2/3)C_1$. Equation 8 can be written the same way. That means any numbers C_1 and C_2 with that particular ratio will solve both equations. So this time we have a useful solution—an infinite number of them, in fact—which are the first normal mode we saw above.

$$\begin{aligned} x_1(t) &= A\cos\left(\sqrt{2}\ t\right) \\ x_2(t) &= (2/3)A\cos\left(\sqrt{2}\ t\right) \end{aligned}$$

Why was it that Equations 6 got us nowhere, but Equations 7 and 8 worked so well? Because the first set of equations had only one solution; the second set of equations was "linearly dependent" (or "redundant"), and had infinitely many solutions.

The "determinant" of a matrix determines if a set of simultaneous equations has only one solution, as opposed to being "linearly dependent" (infinitely many solutions) or "inconsistent" (no solutions).

Finally, what led us to expect that we could find a normal mode solution in the first place? It turns out that in almost every case linear differntial equations for coupled oscillators have normal mode solutions, and the general solutions can be built up from linear combinations of these solutions.

If you've seen linear algebra before, you may remember finding "eigenvectors" and "eigenvalues." The eigenvectors of a matrix tell you the normal modes of a system, and the eigenvalues tell you the frequencies of those normal modes.

Stepping Back

One of our goals in this exercise is to have you understand normal modes as a way of thinking about the behavior of a coupled oscillator. Equations 3 represent a simple, periodic, and *self-perpetuating* solution to the differential equation: that is, if the system is ever in that state, it will stay in that state forever. Equations 4 represent another self-perpetuating solution. Any combination of these solutions is also a solution (because the differential equations are linear and homogeneous). Most remarkably, any state of the system can be represented as a linear combination of these solutions—so once you know the combination, you know just how it will evolve.

But we also want to set up the two key places where matrices play into the process.

- We can use a matrix to convert from A and B to $x_1(t)$ and $x_2(t)$. ("If the first coefficient is 7 and the second is -4, what are the balls actually doing?") As you will see this is a job ideally suited for matrices, because it is a linear transformation from one multi-variable representation to a different multi-variable representation of the same state. An inverse matrix performs the same conversion in the other direction, finding the right combination of normal modes to match any given initial conditions.
- We can also use a matrix to find the normal modes in the first place. The "eigenvalues" of a matrix give the frequency of each normal modes, and the "eigenvectors" give the ratio of x_1 to x_2.

These two uses of matrices are not unrelated, but that won't be clear until you have learned about eigenvalues. Focus now on how two numbers such as "$A = 26/3$ and $B = -11/3$" can describe the entire behavior of the system—the initial conditions and how they will evolve through time—and you will begin your study of linear algebra on a solid foundation.

Problems: The Three-Spring Problem

1. (a) If the three-spring system in the exercise begins at rest with $x_1(0) = 3$ and $x_2(0) = 2$ then it is in the first normal mode. Describe in words how this system will evolve over time.
   (b) If the three-spring system begins at rest with $x_1(0) = 3$ and $x_2(0) = -1$ then it is in the second normal mode. How will the resulting behavior be like Part (a), and how will it be different?
   (c) Now—suppose you solved the differential equations with the initial conditions in Part (a), and then you solved them again with the initial conditions from Part (b), and then you added those two solutions. Would the resulting functions necessarily solve the differential equations? Why or why not? What initial conditions would they represent?
   (d) Describe in words how the system in Part (c) will evolve over time.
2. The general solution to Equations 1–2 with $\dot{x}_1(0) = \dot{x}_2(0) = 0$ is any combination of Equations 3 and Equations 4.
   (a) Write the solution represented by the constants $A = -10$, $B = 0$. What initial conditions $x_1(0)$ and $x_2(0)$ does this solution represent?
   (b) Write the solution represented by the constants $A = 10$, $B = -9$. What initial conditions $x_1(0)$ and $x_2(0)$ does this solution represent?
   (c) What constants A and B lead to the initial conditions $x_1(0) = 13.5$ and $x_2(0) = -3$? What is the solution to the differential equations with these initial conditions?
3. Which of the following represents a normal mode? We are not asking if they are solutions to the differential equations in this exercise, or to any other particular equations. We are simply saying: if some system moved as described by these functions, would that be considered a normal mode of the system? For each one explain in one brief sentence why it is or isn't a normal mode.
   (a) $x_1 = A\sin t$, $x_2 = 2A\sin t$
   (b) $x_1 = A\sin t$, $x_2 = 2A\sin(2t)$

(c) $x_1 = Ae^{2it}$, $x_2 = 3Ae^{2it}$

(d) $x_1 = Ae^{2it}$, $x_2 = Ae^{4it}$

4. Consider a system defined by the positions of two objects, $a(t)$ and $b(t)$. We're not going to tell you anything about what those two objects are, or the differential equations that represent them. We will tell you that the positions oscillate sinusoidally, and they both start at rest at $t = 0$.

   (a) Suppose we tell you that "If $b = 6a$ then both positions will oscillate with a frequency of 10 radians/sec." If you start with $a = 2$ and $b = 12$ and watch the system go for a few seconds, will b still equal $6a$?

   (b) Suppose we tell you that "If $b = 7a$ then a will oscillate at 2 cycles per second and b at 3 cycles per second." If you start with $a = 2$ and $b = 14$ and watch the system go for a few seconds, will b still equal $7a$?

5. Consider a system of balls on springs like the ones we've been discussing in this exercise and assume the system is in one of its normal modes.

   (a) Is it possible for one of the balls to be to the right of its equilibrium point while another one is to the left of its equililbrium point? Why or why not?

   (b) Is it possible for one of the balls to pass through its equilibrium point at a time when another one is not at its equilibrium point? Why or why not?

   (c) How would your answers change if the system were not in a normal mode?

6. In this problem you will consider the three-spring problem with both masses equal and all three spring constants equal. (This is *not* the situation described by Equations 1 and 2.) You should be able to answer all of the following questions by thinking about the physical situation without doing any calculations.

   (a) First consider initial conditions in which you pulled both balls to the right by the same amount and let go.

      i. Spring 1 (the left-most spring) would pull Ball 1 to the left. Spring 3 would push Ball 2 to the left. Which of these forces would be larger, or would they be equal? Why?

      ii. Would the force of Spring 2 on Ball 1 be to the left, to the right, or zero? Why?

      iii. Describe the long term behavior of the system given these initial conditions. Explain why this would be a normal mode solution.

   (b) Describe the other normal mode of the system. Specify both the initial conditions and the long term behavior associated with this second normal mode.

7. *[This problem depends on Problem 6.]* Problem 6 described the initial conditions that would lead to the two normal modes of a coupled system. For example, if you started with the initial conditions $x_1(0) = x_2(0) = 1$ that would put the system in the first normal mode, oscillating with amplitude 1.

   (a) In a similar way, give a set of initial conditions $x_1(0)$ and $x_2(0)$ that would put the system in the second normal mode, with the balls again oscillating with amplitude 1.

   (b) Now suppose the system starts with initial conditions $x_1 = 0$, $x_2 = 1$. Write this initial condition as a combination of the two normal modes you found in Problem 6. Use A and B for the normal modes. Your answer should be in the form $aA + bB$, but with specific numbers for a and b.

   (c) Is the solution in Part (b) a normal mode? Why or why not?

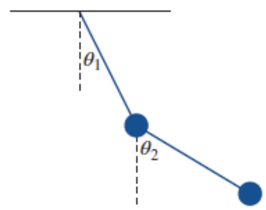
8. *[This problem depends on Problem 6.]* With $k = 8$ N/m and $m = 4$ kg the problem described in Problem 6 is described by the equations $\ddot{x}_1 = -4x_1 + 2x_2$, $\ddot{x}_2 = 2x_1 - 4x_2$.

   (a) Solve these equations numerically with initial conditions corresponding to each of the normal modes you found. Plot the solutions and describe the behavior of x_1 and x_2 in each case.

   (b) Repeat Part (a) with initial conditions corresponding to 2 times one normal mode plus 3 times the other.

   (c) Is the solution you plotted in Part (b) a normal mode solution? Why or why not?

9. The picture shows a "double pendulum": a pendulum hanging from a pendulum. Consider the following possibilities.



   (a) The top pendulum swings with a period of 2 s, and the bottom pendulum swings from the top pendulum with a period of 2 s. Is this a normal mode? Is the resulting motion periodic?

   (b) The top pendulum swings with a period of 2 s, and the bottom pendulum swings from the top pendulum with a period of 3 s. Is this a normal mode? Is the resulting motion periodic?

   (c) The top pendulum swings with a period of 2 s, and the bottom pendulum swings from the top pendulum with a period of π s. Is this a normal mode? Is the resulting motion periodic?

10. In this problem you will derive the equations of motion for the three-spring system. The only physics you need is $F = ma$ and the fact that a spring exerts a force $F = -kx$ where k is the spring constant and x is the displacement.

   (a) The position $x_1 = x_2 = 0$ represents the equilibrium position. Now imagine that Ball 1 is at this position precisely, but Ball 2 is slightly to the right of this position, as shown in Figure 1. Which springs now push on Ball 1, and in which directions? Which springs now push on Ball 2, and in which directions?

   (b) Now imagine that Ball 1 is displaced to the right by a distance x_1, and Ball 2 is displaced to the right by x_2. This time the answers will be quantitative, and they will include the three spring constants k_1, k_2, and k_3. We begin with Ball 1.

      i. The force of Spring 1 on Ball 1 depends only on the position x_1 (the other position is irrelevant). Write a formula for this force.

      ii. How much is Spring 2 stretched? Your answer should be a function of x_1 and x_2. For example, if both balls are displaced the same amount to the right then Spring 2 isn't stretched at all. As a check on your answer, make sure it gives a positive answer when Spring 2 is longer than its equilibrium length, a negative answer when it is shorter, and 0 when Spring 2 is at its equilibrium length (neither stretched nor compressed.)

      iii. Using your answer to Part (ii), write a formula for the force of Spring 2 on Ball 1.

      iv. Putting the two forces together and using $F = ma$, write a differential equation for $x_1(t)$. Be sure you have the sign of each force correct. *Hint:* One way to check yourself is to plug in numbers and make sure you get Equation 1!

   (c) Repeat Part (b) for x_2.

11. Plug Equations 3 into Equations 1 and 2 to confirm that they are valid solutions for any constant A.

12. Our treatment of the three-spring problem was incomplete because we looked only at the cosine parts of the solutions, ignoring the sines.

   (a) Show that the following equations are valid solutions to Equations 1 and 2 for any constants A_1 and A_2.

$$\begin{aligned} x_1(t) &= A_1 \cos\left(\sqrt{2}\,t\right) + A_2 \sin\left(\sqrt{2}\,t\right) \\ x_2(t) &= \frac{2}{3} A_1 \cos\left(\sqrt{2}\,t\right) + \frac{2}{3} A_2 \sin\left(\sqrt{2}\,t\right) \end{aligned} \tag{9}$$

   (b) Show that the initial conditions $\dot{x}_1(0) = \dot{x}_2(0) = 0$ lead to $A_2 = 0$ and therefore to the solution we used in the explanation.

(c) Do Equations 9 represent a normal mode of the system?

(d) The three-spring problem has four initial conditions: the initial position and velocity of Ball 1 and the initial position and velocity of Ball 2. What *must* be true of all the initial conditions for the balls to follow Equations 9?

13. In our second example we created a new solution by combining the two normal modes, along with carefully choosing values of the arbitrary constants.

    (a) Show that Equations 5 are a valid solution to Equations 1 and 2.

    (b) Use Equations 5 to find $x_1(0)$ and $x_2(0)$ and confirm that they meet the intended initial conditions.

14. Arguably the most important point in our treatment of the three-spring problem—both for understanding normal modes, and for setting up linear algebra—is creating new solutions as linear combinations of the two solutions we found. (This works for any linear homogeneous differential equation.)

    (a) Show that if you add Equations 3 to Equations 4 the resulting functions are valid solutions to Equations 1 and 2 for any values of A and B.

    (b) What are the initial positions $x_1(0)$ and $x_2(0)$ for these generalized solutions? Your answers will be functions of A and B.

    (c) Now turn it around; starting with your answers to Part (b), solve for A and B as functions of $x_1(0)$ and $x_2(0)$. In doing so, you find a direct way to meet any given initial positions, as we did in our third example. (Note that we still do not have a fully general solution to the three-spring problem. We left off all the sine terms based on our initial assumption that both balls start at rest, so we don't have to worry about $v_1(0)$ and $v_2(0)$.)

    (d) Write the solution if you pull Ball 1 to the left by a distance of 1 and Ball 2 to the right by the same amount and release them.

15. A key point in our explanation is that a normal mode represents simple oscillation with one frequency, but that *combinations* of normal modes are generally not simple or periodic. Often they do not appear to be built from sines and cosines (even though they are). Graph each function below from $t = 0$ to $t = 30$.

    (a) $\sin\left(\sqrt{2}\,t\right)$.

    (b) $\sin\left(\sqrt{5}\,t\right)$.

    (c) $\sin\left(\sqrt{5}\,t\right) + \sin\left(\sqrt{2}\,t\right)$.

    (d) $3\sin\left(\sqrt{5}\,t\right) + 2\sin\left(\sqrt{2}\,t\right)$.

16. In this problem you will solve Equations 1 and 2 by guess and check.

    (a) Plug in the solution $x_1 = C_1\cos(2t)$, $x_2 = C_2\cos(2t)$. Write down and solve the resulting two linear equations for C_1 and C_2. What motion does the resulting solution describe?

    (b) Plug in the solution $x_1 = C_1\cos\left(\sqrt{3}\,t\right)$, $x_2 = C_2\cos\left(\sqrt{3}\,t\right)$. Write down and solve the resulting two linear equations for C_1 and C_2. What motion does the resulting solution describe?

    (c) Plug in the solution $x_1 = C_1\cos(\omega t)$, $x_2 = C_2\cos(\omega t)$ where ω is a constant. Show that for any value of ω, you can solve the resulting linear equations with $C_1 = C_2 = 0$.

    (d) Now plug in the solution $x_1 = C_1\cos\left(\sqrt{5}\,t\right)$, $x_2 = C_2\cos\left(\sqrt{5}\,t\right)$. Show that the resulting linear equations both reduce to the *same* equation, and are therefore solved as long as C_1 and C_2 are in the correct ratio. This ratio should lead you to our second normal mode.

17. In this problem you will walk through the entire three-spring problem, from the beginning, with different masses and spring constants. The differential equations you will solve are:

$$\frac{d^2x_1}{dt^2} = -5x_1 + 8x_2 \tag{10}$$

$$\frac{d^2x_2}{dt^2} = x_1 - 3x_2 \tag{11}$$

(a) The functions $x_1 = A\cos t$, $x_2 = kA\cos t$ provide a valid solution to Equations 10 and 11 for *any* value of A, provided you choose the right value of k. Show that this works by finding the right value of k.

(b) Show that the functions $x_1 = C\cos(2t)$, $x_2 = kC\cos(2t)$ do not provide a valid solution unless $C = 0$.

(c) Show that the functions $x_1 = D\cos\left(\sqrt{2}\,t\right)$, $x_2 = kD\cos\left(\sqrt{2}\,t\right)$ do not provide a valid solution unless $D = 0$.

(d) The functions $x_1 = B\cos\left(\sqrt{7}\,t\right)$, $x_2 = kB\cos\left(\sqrt{7}\,t\right)$ provide a valid solution to Equations 10 and 11 for *any* value of B, provided you choose the right value of k. Show that this works by finding the right value of k.

(e) Parts (a) and (d) represent solutions, but are they normal modes? How can you tell?

(f) What initial conditions are represented by the solutions to Parts (a) and (d)?

(g) Find a solution to match the initial conditions $x_1 = -2$, $x_2 = 2$.

(h) Find a solution to match the initial conditions $x_1 = 10$, $x_2 = 2$.

18. Each of the scenarios below refers to a solution to Equations 1–2. For each one have a computer create an animation of two moving balls. Draw them as balls oscillating about their equilibrium positions, which you should separate by a distance of 3.

(a) Show the two balls oscillating in the first normal mode, with $A = 2$.

(b) Show the two balls oscillating in the second normal mode, with $B = 2$.

(c) Show the two balls oscillating in a sum of the two normal modes with $A = B = 1$.

(d) Describe the motion in each of the three cases.