Computer Problems for Linear Algebra

1. One possible basis for the xy plane consists of the vectors $\vec{S}_1 = \hat{i} + 2\hat{j}$ and $\vec{S}_2 = 2\hat{i} + 3\hat{j}$. The transformation $\mathbf{F}$ is represented in the $\hat{i}\hat{j}$ basis by the matrix $\mathbf{F_i} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and in the $\vec{S}_1\vec{S}_2$ basis by the matrix $\mathbf{F_s} = \begin{pmatrix} -7 & -11 \\ 5 & 8 \end{pmatrix}$.
   (a) Begin with the vector $\vec{U} = \vec{S}_1 + 2\vec{S}_2$.
      i. Calculate $\mathbf{F_s}\vec{U}$ (working entirely in the $\vec{S}_1\vec{S}_2$ basis).
      ii. Draw $\vec{S}_1\vec{S}_2$ graph paper. On that paper show $\vec{U}$ and $\mathbf{F_s}\vec{U}$.
      iii. Now, use the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ to convert vector $\vec{U}$ to the $\hat{i}\hat{j}$ basis.
      iv. Act on your new vector with $\mathbf{F_i}$. Draw the $\hat{i}\hat{j}$ vectors before and after. (Your picture should look exactly like your previous picture, showing that the same vector has been acted on by the same transformation.)
   (b) Repeat Part (a) on a different vector of your own choosing. Remember to start by choosing a vector in the $\vec{S}_1\vec{S}_2$ basis.
2. The matrix $\mathbf{N} = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 1 & 0 \end{pmatrix}$ represents the shape formed by a line from $(0, 0)$ to $(0, 1)$, a line from $(0, 1)$ to $(1, 0)$, and so on. We will call this shape Nemo.
   (a) Draw Nemo.
   (b) Find the matrix $(2/3)\mathbf{N}$ and draw the resulting shape. Look! Nemo swam further away!
   (c) Matrix $\mathbf{R} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$ represents a transformation. Find matrix $\mathbf{RN}$ and draw the resulting shape. What did Nemo do this time?
   (d) Matrix $\mathbf{T} = (2/3)\mathbf{R}$ is a single matrix that performs both of these transformations. Compute this matrix and apply it to matrix $\mathbf{N}$.
   (e) Now apply matrix T again to your *answer* to part (d). Then apply matrix $\mathbf{T}$ again to that answer. Repeate the process 10 times and draw the matrices $\mathbf{N}$, $\mathbf{TN}$, $\mathbf{TTN}$, $\ldots \mathbf{T}^{10}\mathbf{N}$ together on one plot (or better yet in an animation) to see Nemo slowly spiral down into a whirlpool.
3. Use inverse matrices to solve the equations $9a + 3b - 5c + 7d = 8$, $3a - 4b + c = 10$, $10a + 3b + c + 6d = 3$, $a - b + 2c - d = 69$. (You can have a computer solve these equations to check your answer, but to solve the problem you need to show the steps of using an inverse matrix to solve them, using the computer to do the matrix algebra for you.)

In Problems 4–5 you will be given a set of equations.

- Classify the set of equations as "all homogeneous" or "not all homogeneous"
- If the equations are all homogeneous, classify them as having a unique solution, or as being linearly dependent (infinitely many solutions).
- If the equations are not all homogeneous, classify them as having a unique solution, or as being "inconsistent or linearly dependent.'

4. $-6x + 3y + 4z - 2q + 5r = 2$, $-x + 2y - z + q + 2r = -3$, $-4x + 2y + 3z - 2q - r = -1$, $2x - 2y - z + 2q + r = 0$, $5x - 5y - 3q = 2$
5. $x + 2y + 3z + 4q + r = 0$, $5x + 6y + 7z + 8r = 0$, $9x + 10y + 11q + 12r = 0$, $13x + 14z + 15q + 16r = 0$, $17y + 18z + 19q + 20r = 0$

For Problems 6–7 find the value(s) of λ for which the given homogeneous equations have nontrivial solutions.

6. $(2-\lambda)x + 3y - 2z = 0,\ -x + (1-\lambda)y + 2z = 0,\ 3x + 3y + (-1-\lambda)z = 0$

7. $(1-\lambda)x + 2y + 3z + 4q + r = 0,\ 5x + (6-\lambda)y + 7z + 8r = 0,\ 9x + 10y - \lambda z + 11q + 12r = 0,\ 13x + 14z + (15-\lambda)q + 16r = 0,\ 17y + 18z + 19q + (20-\lambda)r = 0$

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8. For each set of vectors shown below, say whether it can form a basis, and how you know. (The answer in each case only needs to be a few words long.)

   (a) $\vec{V}_1 =< 1,2,3 >,\ \vec{V}_2 =< 4,5,6 >,\ \vec{V}_3 =< 1,0,1 >$

   (b) $\vec{V}_1 =< 1,2,3,4 >,\ \vec{V}_2 =< -1,3,-2,0 >,\ \vec{V}_3 =< 2,9,7,12 >, \vec{V}_4 =< -2,3,-1,2 >$

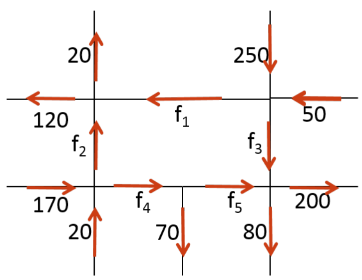
   (c) $\vec{V}_1 =< 1,2,0,-1,3 >,\ \vec{V}_2 =< 1,3,0,0,2 >,\ \vec{V}_3 =< 2,-1,3,4,1 >, \vec{V}_4 =< 6,-3,0,2,1 >$

9. Matrix $\mathbf{M} = \begin{pmatrix} 3 & -2 & 6 & 5 \\ 1 & 4 & 0 & 7 \\ 8 & 1 & 2 & -4 \\ -1 & -1 & 10 & 21 \end{pmatrix}$.

   (a) Find $|\mathbf{M}|$.

   (b) Find $\mathbf{M}^{-1}$.

10. The diagram below shows a map of one way streets. The numbers and letters represent known and unknown rates of traffic flow, measured in cars per hour.



   (a) For each intersection, write an equation that says "The total rate of cars coming in equals the total rate of cars going out."

   (b) Is there a unique solution for all of the unknown flow rates? Explain how you got your answer, including what you asked a computer or calculator to calculate for you and what answer it gave.

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

11. **Exploration: A Quantum Mechanical Well.** Many quantum mechanics problems start by solving Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

The potential function $V(x)$ is specified in the problem (just as classical dynamics problems begin by specifying a force). In this problem you will solve Schrödinger's equation for the potential function:

$$V=\begin{cases} V_0 & x<0 \\ 0 & 0\le x\le a \\ V_0 & x>a \end{cases}$$

where m, $\hbar$, V_0, a and E are positive constants, and (this is very important) $E<V_0$. The value of ψ is generally complex, but in this problem make all your solutions real. The boundary condition is that $\lim_{x\to\pm\infty}\psi(x)=0$.

(a) Begin by writing and solving Schrödinger's equation in the right-most region. Your general solution will have two arbitrary constants, but the boundary condition at $x\to\infty$ will eliminate one of them.

(b) Repeat Part (a) for the left-most region. Once again, your final solution will have only one arbitrary constant (but it is not the *same* arbitrary constant as in the first solution).

(c) Write and solve Schrödinger's equation in the middle region. This time you will be left with two arbitrary constants.

You now have four arbitrary constants: one on the left, one on the right, and two in the middle. But now we introduce a postulate of quantum mechanics: both $\psi(x)$ and $d\psi/dx$ must be continuous. So $\psi(a)$ calculated from Part (a) must agree with $\psi(a)$ calculated from Part (c) and so on.

(d) The requirement of continuity imposes four different restrictions, two on each boundary. Write the equations that represent those restrictions. *Hint*: you can save some writing if you define two new constants: $\alpha=\sqrt{2m(V_0-E)}/\hbar$ and $\beta=\sqrt{2mE}/\hbar$.

(e) You should now have four homogeneous linear equations for four arbitrary constants. One solution is the trivial one where they are all zero, but that can't represent a physical state, so the only possible physical states are ones for which their are other solutions. Write an equation that must be satisfied in order for those nontrivial solutions to exist. This will require taking a 4×4 determinant, but it will have enough zeroes in it that expansion by minors will not be too time consuming to do by hand. Simplify the equation as much as possible (leaving it in terms of α and β). This equation describes the physically possible energies E for a particle in a finite potential well, although it cannot be analytically solved for E.

(f) At what step would your answers first have begun to look different if E were greater than V_0?

12. *[This problem depends on Problem 11.]* The mass of an electron is 9.1×10^{-31} kg and the constant $\hbar$ is 6.63×10^{-34} J·s. Consider an electron in a potential well with $V_0=10^{-18}$ J and $a=10^{-8}$ m.

(a) Write the function $f(E)$ that must equal 0 at a physically allowed value of the energy.

(b) Plot that function from $E=0$ to $E=V_0$. How many allowed values of energy are there for the electron in this well?

(c) Numerically find the value of the lowest allowed energy. (It must be positive.) Using these numbers your answer will come out in Joules. Convert them to the more standard particle physics energy unit of "electron volts": 1 eV=1.6×10^{-19} J.

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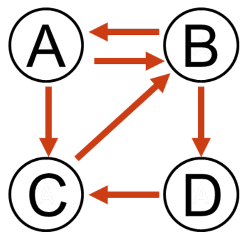
13. **Exploration: Google PageRank**

One of the techniques Google uses to select search results is "PageRank," invented by Google founders Sergey Brin and Larry Page (for whom it is named).[1] The basic idea is that each web page is given a rank based on what other pages link to it. The higher a page's rank is, the more rank it confers on other pages that it links to. To illustrate how the system works, consider the following web of four pages.



Initially each page is given a ranking of 1 over the number of pages, so in this case they each start with 1/4. In each iteration, each page distributes its current rank equally among all the other pages that it links to. For example, page A links to pages B and C, so it gives them each half of its current rank, or 1/8. Page C only links to page B, so it gives its entire current rank to B. After the first iteration, page B has a new rank of 3/8, the sum of the ranks it inherited from A and C. (It doesn't matter if page A links to page B once or twenty times; the algorithm only counts *whether* one page links to another. It also ignores any links from a page to itself.)

(a) Write the column vector $\mathbf{r}_1$ representing the rankings of the four pages after one iteration. Write the vector $\mathbf{r}_2$ giving their rankings after two iterations.

(b) Write the matrix $\mathbf{L}$ that you multiply by $\mathbf{r}_i$ to get $\mathbf{r}_{i+1}$.

(c) In the limit of infinitely many iterations, you approach a vector $\mathbf{r}$ that is no longer changing. From this you can conclude that $\mathbf{r}$ is an eigenvector of $\mathbf{L}$. What is its eigenvalue?

(d) Find the solution $\mathbf{r}$ for this particular web.

PageRank has a simple interpretation. If a user starts on a random page and randomly follows links, the rank of a given page after i iterations is the probability that he will be on that page after following i links. In practice, however, users sometimes jump to a new random page rather than following links. If d is the probability of a user following a link, and $1-d$ is the probability of the user jumping to a new random page, then the probability of landing on the n^{th} page after i iterations is given by:

$$r_i(n) = \frac{1-d}{N} + d\sum_{m=1}^{N} r_{i-1}(m)L_{mn}$$

Here $r_i(n)$ is the rank of page n after i iterations, and N is the total number of pages. That may sound complicated, but it's just what you did above. If a page links to seven other pages, then each iteration it gives 1/7 of its rank to each of those pages. The difference is that now it gives $d/7$ of its rank to each of those pages, and each page also receives a rank $(1-d)/N$ for the chance that the user jumped to that page randomly instead of following a link. This formula can be written in matrix form, using $\mathbf{1}$ for a column matrix where all the entries are 1.

$$\mathbf{r}_i = \left(\frac{1-d}{N}\right)\mathbf{1} + d\,\mathbf{L}\,\mathbf{r}_{i-1} \tag{0.1}$$

The probability d is called the "damping factor."

[1] Sergey Brin and Lawrence Page. 1998. The anatomy of a large-scale hypertextual Web search engine. Comput. Netw. ISDN Syst. 30, 1-7 (April 1998), 107-117.

(e) Show that in the case $d = 1$ Equation 0.1 reduces to the simpler formula you were using above. What assumption does $d = 1$ represent?

(f) Once again the steady state solution is the one where $\mathbf{r}$ doesn't change from one iteration to the next. Write a matrix equation for $\mathbf{r}$ and solve it. (In this part you're considering a general web, not the particular four-page example given above.) *Big hint*: Solving for $\mathbf{r}$ will require bringing both the $\mathbf{r}$ terms to one side of the equation. To combine those two terms you'll need to insert an identity matrix $\mathbf{I}$ in front of one of them. Finally, to get $\mathbf{r}$ by itself you'll multiply both sides by the inverse of the matrix that multiplies it.

(g) What would you expect the rankings to approach in the limit $d \to 0$ and why? What would you expect them to approach in the limit $d \to 1$ and why?

14. *[This problem depends on Problem 13.]* Unless otherwise specified, everything in this problem refers to the four-page web given in Problem 13.

(a) Find the solution $\mathbf{r}$ for the four-page web given above, using a damping factor $d = 0.85$ (which is the value recommended by Brin and Page). How do the numbers look different from the undamped solution you found above, and why do these differences make sense in light of your answers to Part (g)?

(b) Using the simple algorithm with no damping factor, calculate the first 100 iterations of $\mathbf{r}_i$. Make a plot showing each of the four ranks as a function of i. For reference, put horizontal lines on your plot representing the steady state values $\mathbf{r}$.

(c) Repeat Part (b) using a damping factor $d = 0.5$. (The horizontal lines in this plot should reflect the damped solution.)

(d) Calculate the steady state solution $\mathbf{r}$ for 100 values of d from 0 to 1. (You may find it easier to include values close to 1, but not $d = 1$ itself.) Make a plot showing the steady-state solution for each of the four ranks as a function of d. Explain what your plot looks like and why it makes sense.

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15. Find the eigenvectors and eigenvalues of the following matrix and describe in words what transformation this matrix performs: $\begin{pmatrix} 4/3 & 1/3 & 1/3 \\ 1/3 & 4/3 & 1/3 \\ 1/3 & 1/3 & 4/3 \end{pmatrix}$

16. 3D objects are often represented on computers by lists of polygons that can be drawn to approximate the surface of the shape. In this problem you're going to represent and manipulate a sphere.

(a) Define a set of polygons representing a sphere with 8 lines of longitude and 8 lines of latitude. The first set should be a set of triangles going from the point $(0, 0, 1)$ points with $\theta = \pi/8$ and ϕ going from 0 to 2π in 8 steps. Each triangle will use two different values of ϕ. The next polygons should be quadrilaterals connecting points with $\pi/8$ to points with $\theta = \pi/4$, again with two values of ϕ per polygon. Continue until you get a set of triangles that reach the south pole. Have the computer draw all these polygons.

(b) Redefine the list and redraw the sphere with ϕ and θ taking on 64 values each instead of 8.

(c) Write a matrix $\mathbf{S}$ that stretches vectors by a factor of 2 in the y direction. Apply S to each polygon vertex and draw the result. What does it look like?

(d) Define a matrix $\mathbf{R}$ that rotates vectors 45° around the z axis. Apply R to the vertices of the original polygons and draw the result. What does it look like?

(e) Apply the matrix $\mathbf{RS}$ to the original vertices and redraw the shape. Explain why it looks the way it does.

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17. **Exploration: Polynomial Approximation.** The "Legendre polynomials" $P_n(x)$ can be defined in several different ways, but one common form of them defines them so they are orthonormal: that is, $\int_{-1}^{1} P_n(x)P_m(x)dx$ is 0 if $n \neq m$ and 1 if $n = m$. You don't need to have heard of Legendre polynomials before to do this problem.

    Consider a function $f(x)$. Your goal is to find the third order polynomial $\hat{f}(x)$ that best approximates $f(x)$ between $x = -1$ and $x = 1$. Our definition of "best approximation" will be the function that minimizes $\int_{-1}^{1}\left[f(x) - \hat{f}(x)\right]^2 dx$. A Taylor series is the best polynomial approximation at and around a given point, but this technique can give you a better approximation on a given interval.

    For reasons that will soon be clear, we will look for an answer in the form $\hat{f}(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x)$. Your goal is to find the constants c_0–c_3.

    (a) Write the integral that represents the quantity you are trying to minimize.

    (b) Your integrand is the square of a 5-term series. We're not asking you to write all 25 resulting terms, but write the term that involves $[P_2(x)]^2$.

    (c) Now write the term that involves $P_2(x)P_3(x)$.

    (d) Evaluate those two integrals. Remember that the Legendre polynomials are orthonormal!

    (e) Now that you see the pattern, expand the square. All the integrals that involve only Legendre polynomials can be replaced with 0 or 1, as appropriate, leaving only integrals that involve $f(x)$.

    (f) Two of the terms in your answer involve c_0. Find the value of c_0 that minimizes those two terms. (Your answer will have the functions $f(x)$ and $P_0(x)$ in it, and you can just leave them like that.)

18. *[This problem depends on Problem 17.]* The first four normalized Legendre polynomials are $P_0 = 1/\sqrt{2}$, $P_1 = \sqrt{3/2}\,x$, $P_2 = (1/2)\sqrt{5/2}(3x^2 - 1)$, and $P_3 = (1/2)\sqrt{7/2}(5x^3 - 3x)$.

    (a) Find the best third order polynomial approximation to e^x between $x = 1$ and $x = -1$. (This will involve either computer integration or some tedious integration by parts. The exact coefficients will be complicated expressions, but you can just give numerical values for them.)

    (b) Show on one plot the original function e^x between $x = -1$ and $x = 1$ and each of the first four partial sums of its polynomial expansion. (The first partial sum is the zero order approximation and the fourth partial sum is the third order approximation.)

    (c) Find the fourth order Maclaurin series for e^x.

    (d) Show on one plot the original function e^x between $x = -1$ and $x = 1$, the second order approximation you derived using Legendre polynomials, and the second order Maclaurin series. Which approximation looks better?

    (e) Show on one plot the original function e^x between $x = -0.1$ and $x = 0.1$, the second order approximation you derived using Legendre polynomials, and the second order Maclaurin series. Which approximation looks better?

    (f) In what kinds of situations would each of these approximations be most useful?

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19. *True story*: A chemical engineering professor we know was once creating a homework problem involving a combustion reactor in which methane (CH_4) and ethane (C_2H_6) react with oxygen (O_2) to form carbon monoxide (CO), carbon dioxide (CO_2), and water (H_2O).

$$CH_4 + \frac{3}{2}O_2 \rightarrow CO + 2H_2O$$
$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$
$$C_2H_6 + \frac{5}{2}O_2 \rightarrow 2CO + 3H_2O$$
$$C_2H_6 + \frac{7}{2}O_2 \rightarrow 2CO_2 + 3H_2O$$

This reaction involves six molecular species. The initial concentrations of these species would be given as part of the problem. Of the six final concentrations, some would be given as "measured values" and the students would calculate the rest. According to the method he was using, the number of measured values he would have to give equals the number of species minus the number of reactions. In this example he would specify two final concentrations $(6 - 4)$ and the students would calculate the other four. But his calculations did not lead to consistent answers. In this problem you will see why, and how many final concentrations he needed to specify in order to make his problem work.

(a) Let A represent the final concentration of CH_4, B the final concentration of C_2H_6, and C, D, E, and F the final concentrations of O_2, CO, CO_2, and H_2O, respectively. Write an algebraic equation to represent each chemical reaction by replacing each chemical species with the symbol for its concentration and the arrow with an equal sign. For example, the first reaction equation would be $A + (3/2)C = D + 2F$.

(b) Consider A and B to be "constants" and the remaining concentrations as variables. (We could just as well have chosen any other two variables.) Rearrange your equations into standard form, with the variables on the left and the constants on the right. Simplify the equations so that all coefficients are integers.

(c) Suppose you determined, using either a determinant or row reduction, that your four equations have a unique solution. What would that tell you about the reaction? That is, what would you measure and what could you then calculate?

(d) Write an augmented matrix for your four equations and reduce it to row echelon form.

(e) Mathematically—forget about chemistry for a moment—what does your final matrix tell you about the equations you started with?

(f) Now let's return to our chemical engineer. He will have to measure the concentrations of some of his species, and then he will be able to calculate the rest. How many does he need to measure, and how many can he then calculate?

(g) We found in this case that the rows of the augmented matrix were linearly dependent, and that in turn told us something about the reaction. What different lesson would we draw if the rows of the augmented matrix were inconsistent?

20.  *[This problem depends on Problem 19.]* An experiment involves the following five reactions. The initial concentrations of all six species are known. You are going to measure as many final concentrations as you have to, and then calculate the remaining concentrations based on those measurements. How many concentrations do you have to measure, and how many can you then calculate?

$$2N_2O_5 \rightarrow 2N_2O_4 + O_2$$
$$N_2O_5 \rightarrow NO_2 + NO_3$$
$$NO_3 \rightarrow O_2 + NO$$
$$NO + N_2O_5 \rightarrow 3NO_2$$
$$2NO_2 \rightarrow N_2O_4$$

You may have a computer do the row reduction; your job is to set up the matrix and then interpret the results.

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

21. **Inertia Tensor** In introductory physics you probably learned that a body has a "moment of inertia" I about any possible axis, which plays a role analogous to mass for linear motion. You may not have been taught that $\mathbf{I}$ is a tensor with nine components. (In this problem and the next we will use the symbol $\mathbf{I}$ for the inertia tensor—no relation to the identity matrix.) For a continuous object of density ρ, $I_{ij} = \int_V \rho \left[\delta_{ij} \left(\sum_k x_k^2\right) - x_i x_j\right] dV$, where $\int_V dV$ means a triple integral over the volume, and x_1, x_2, and x_3 are the three Cartesian coordinates x, y, and z. The "Kronecker delta" δ_{ij} equals 1 if $i = j$ and 0 otherwise, so the sum over k only appears in the diagonal elements of $\mathbf{I}$.

    (a) Calculate the inertia tensor of a uniform cube of mass M with corners at the origin and the point (L, L, L).

    (b) Angular momentum is given by the equation $\vec{L} = \mathbf{I}\vec{\omega}$. If the cube in Part (a) moves with angular velocity $\omega = a\hat{i} + b\hat{j}$ where a and b are constants, calculate its angular momentum.

22. *[This problem depends on Problem 21.]*

    (a) The angular equivalent of Newton's second law is $\vec{\tau} = \mathbf{I}\vec{\alpha}$. Rewrite this equation to give $\vec{\alpha}$ as a function of $\vec{\tau}$ and then answer the equation: if a torque $\tau = \tau_0\hat{k}$ is applied to the cube in Problem 21, find the angular acceleration α of the cube.

    (b) Find the inertia tensor of the same cube about a set of axes where the z axis is unchanged and the x axis goes diagonally through the bottom face of the cube. *Hint*: this can be done more easily with a rotation matrix than by integrating all over again.

    (c) The "principal axes" of a body are the ones for which the inertia tensor is diagonal. Find the principal axes of this cube. (The origin remains unchanged at the corner of the cube. If the origin were in the center the principal axes could be guessed from symmetry.)

    (d) If you apply a torque about the first of the principal axes you found, in what direction will the cube rotate?

The Simplex Method

One of the important applications of linear algebra that we teach is the Simplex method for linear programming problems. Problems 23–26 all relate to the Simplex method.

23. **The Transportation Problem** Your warehouse in Atlanta has 300,000 nails, the one in New York has 200,000 and the one in Boston has 500,000. The stores in Chicago, St. Louis, and Louisville need 400,000, 300,000, and 300,000 nails respectively.

    (a) Write, but do not solve, a simplex tableau to answer the question: how many nails should each warehouse send to each store? Assume the transportation cost is proportional to the distance, which is given below.

	Atlanta	New York	Boston
Chicago	700	800	1000
St. Louis	500	1000	1200
Louisville	400	800	1000

    (b) How many nails should each warehouse send to each store?

24. **The Assignment Problem**[2] Suppose you have three workers that each need to be assigned to a job. The workers have different levels of skill and experience; the cost of employing each one to do each job is given below.

	Job 1	Job 2	Job 3
Worker 1	1	3	2
Worker 2	2	3	5
Worker 3	2	2	4

    (a) Define a set of variables x_{ij}, equal to one if worker i has job j, and zero otherwise. Write, but do not solve, a simplex tableau to minimize the total cost subject to the constraints that each worker has exactly one job and each job has exactly one worker.

    (b) Which worker should have each job?

25. **The Diet Problem** The company Sumptuous Land Of Plenty has just been awarded a contract to make stew for school lunches. Each serving must have between 600 and 700 calories, no more than 710 mg of sodium, and no more than 10 g of sugar.[3] The company has hired you to create the recipe that meets these requirements for the least possible cost, using the ingredients.

	Cost per serving ($)	**Calories**	**Sodium (mg)**	**Sugar (g)**
Peanut butter	0.50	94	73	1.5
Salted Lentils	1	230	240	2
Spam	0.20	174	770	0
Soylent green	0.75	400	100	1

    (a) Write, but do not solve, a simplex tableau to answer the question: how many servings of each ingredient should go in the recipe?

    (b) How many servings of each ingredient should go in the recipe?

[2] The simplex method is not the most efficient way to solve the assignment problem for large numbers of variables. See e.g. H.W. Kuhn, The Hungarian Method for the Assignment Problem, Naval Research Logistics Quarterly 2 (1955) 8397.

[3] The calorie and sodium requirements are from the "National School Lunch Program" guidelines for 6^{th}–8^{th} grade, and the nutrition information for peanut butter and spam comes from nutrition Web sites. Everything else in the problem we just made up.

26. **Exploration: Moving Sand** In 1781 Gaspard Monge published his work on the "soil transport" problem: how to most efficiently move a pile of of dirt from a given starting shape to a given final shape. This problem has applications in many areas, including calculating distance between states of a quantum mechanical system.[4] As an example, suppose you have a square sandbox that goes from the origin to the point $(5, 5)$. Initially the height of the sand is given by $S_0 = e^{-x^2-y^2}$. You want to move it to a different corner: $S_f = e^{-(5-x)^2-y^2}$. The cost of moving a unit volume of sand is equal to the distance you move it. In this form the problem involves integrals over the initial and final distributions, but you can turn it into a linear programming problem by breaking the grid into discrete boxes. If each box is 1×1 then you'll have a total of 25 boxes B_{ij}, where i and j each go from 0 to 4.

   (a) Make tables of the average values of S_0 and S_f in each of the 25 boxes. *Hint*: once you calculate this for S_0 you can get it for S_1 from symmetry.

   (b) Your independent variables are x_{ijkl}, representing the amount of sand moved from B_{ij} to B_{kl}. Define an objective function representing the total cost of movement as a function of these variables. Recall that cost is equal to distance times amount of sand. Naively you have 5^4 independent variables, but you can significantly reduce that number by tossing out any term that involves moving sand from a location where $S_f \geq S_0$ or moving sand to a location where $S_0 \geq S_f$.

   (c) Define a set of constraints that reflects the fact that the amount of sand taken from each box equals $S_0 - S_f$ (provided this is positive).

   (d) Define a set of constraints that reflects the fact that the amount of sand moved to each box equals $S_f - S_0$ (provided this is positive).

   (e) Either write a simplex program or use an existing one to find the minimum cost to move the sand.

   (f) Look at the values of the x_{ijkl} in your final solution and describe in words how the sand was moved. You should find that the answer was predictable.

   (g) Find the minimal sand-moving cost for the same initial sand distribution, but with $S_f = (1/2)e^{-(x^2+y^2)/4}$

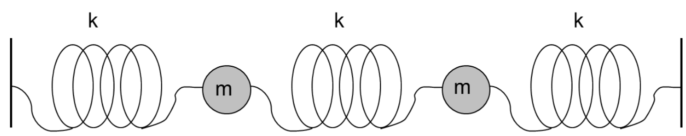
[4] Karol Zyczkowskiyx and Wojeciech Slomczynskiz, "The Monge Distance Between Quantal States," J. Phys. A: Math. Gen. 31 (1998) 9095–9104.

Problems About Coupled ODEs and Normal Modes

When we teach linear algebra we build the unit in large part around solving for the motion of coupled oscillators. That provides motivation and examples of using matrices to change bases, determinants, eigenvalues and eigenvectors, and most other basic linear algebra topics. Problems 27–33 are all related to coupled oscillators or coupled ODEs.

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

27. In this problem you will consider the three-spring problem shown below. You should be able to answer all of the following questions by thinking about the physical situation without doing any calculations.



   (a) First consider initial conditions in which you pulled both balls to the right by the same amount and let go.
      i. Spring 1 (the left-most spring) would pull Ball 1 to the left. Spring 3 would push Ball 2 to the left. Which of these forces would be larger, or would they be equal? Why?
      ii. Would the force of Spring 2 on Ball 1 be to the left, to the right, or zero? Why?
      iii. Describe the long term behavior of the system given these initial conditions. Explain why this would be a normal mode solution.
   (b) Describe the other normal mode of the system. Specify both the initial conditions and the long term behavior associated with this second normal mode.

28. *[This problem depends on Problem 27.]* With $k = 8$ N/m and $m = 4$ kg the problem described in Problem 27 is described by the equations $\ddot{x}_1 = -4x_1 + 2x_2$, $\ddot{x}_2 = 2x_1 - 4x_2$.
   (a) Solve these equations numerically with initial conditions corresponding to each of the normal modes you found. Plot the solutions and describe the behavior of x_1 and x_2 in each case.
   (b) Repeat Part (a) with initial conditions corresponding to 2 times one normal mode plus 3 times the other.
   (c) Is the solution you plotted in Part (b) a normal mode solution? Why or why not?

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29. A normal mode represents simple oscillation with one frequency, but *combinations* of normal modes are generally not simple or periodic. Often they do not appear to be built from sines and cosines (even though they are). Graph each function below from $t = 0$ to $t = 30$.
   (a) $\sin\left(\sqrt{2}\ t\right)$.
   (b) $\sin\left(\sqrt{5}\ t\right)$.
   (c) $\sin\left(\sqrt{5}\ t\right) + \sin\left(\sqrt{2}\ t\right)$.
   (d) $3\sin\left(\sqrt{5}\ t\right) + 2\sin\left(\sqrt{2}\ t\right)$.

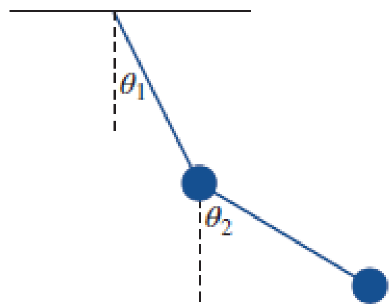
In Problems 30–31 find the normal mode solutions for the given sets of equations. Then find the amplitude of each normal mode if the system starts with the given initial conditions. When writing the solutions you should include all arbitrary constants, but when you plug in initial conditions for second order equations assume the initial first derivatives are zero. The only step that should require a computer is finding the eigenvalues and eigenvectors of matrices, which would be tedious by hand.

30. $\ddot{x} = 3x + y + z$, $\ddot{y} = x + y + 3z$, $\ddot{z} = -x + 3y + z$, $x(0) = 1$, $y(0) = 2$, $z(0) = 3$ *Hint:* For positive eigenvalues you can solve the ODEs using exponentials, but it's easier to apply the condition that the initial derivatives are zero if you write the solution in terms of sinh and cosh. If you have no idea what those are just go ahead and use exponentials and it won't be too bad, and you can learn about sinh and cosh in our chapter on special functions.

31. $\dot{x} = -2x - y + z$, $\dot{y} = -2x + y - 3z$, $\dot{z} = x + y - 2z$, $x(0) = 1$, $y(0) = 1$, $z(0) = 1$. *Notice that these are first derivatives.*

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

32. A double pendulum is shown below.



The equations describing this system depend on the angles θ_1 and θ_2, the gravitational acceleration g, and the length L of the strings.

$$2\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + 2\frac{g}{L}\sin\theta_1 = 0$$
$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{g}{L}\sin\theta_2 = 0$$

These equations are nonlinear. For small oscillations, however, you can assume θ_1, θ_2, and all of their derivatives remain small, and thus approximate this with a set of linear equations.

(a) Replace all the trig functions with the linear terms of their Maclaurin series expansions. Then eliminate any remaining nonlinear terms. The result should be two coupled, linear differential equations.

(b) Solve the equations algebraically for $\ddot{\theta}_1$ and $\ddot{\theta}_2$.

(c) Find the normal mode frequencies of this system for small oscillations.

33. *[This problem depends on Problem 32.]* For this problem take $g = 9.8$ m/s and $L = 1.0$ m and assume in all cases that the two pendulums start at rest.

(a) Choose one of the two normal modes of the system and numerically solve the original equations (not the linearized ones) for $\theta_1(0) = 0.1$, with $\theta_2(0)$ chosen to be whatever it needs to be for that normal mode. Plot $\theta_1(t)$ and $\theta_2(t)$ together on one plot. Include enough time on your plot to see at least 3 full oscillations of the pendulums.

(b) Repeat Part (a) for values of $\theta_1(0) = 0.2$, 0.3, and so on up to 1. In each case adjust $\theta_2(0)$ to whatever the normal mode solution says it should be.

(c) Describe what the system's behavior looks like for small and large amplitudes using these initial conditions. Explain why the different behavior in the different cases makes sense.