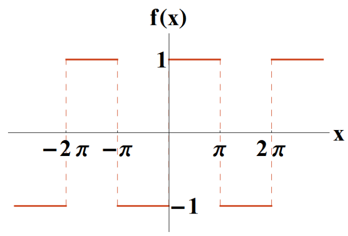
Computer Problems for Fourier Series and Transforms

The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.

1. "Square waves" are frequently used in electronics and signal processing. An example is shown below.



$$y(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ -1 & \pi < x < 2\pi \\ \ldots & \text{and so on} \end{cases}$$

   (a) Write an integral for a_0, the constant term in the Fourier series expansion. You should be able to evaluate this integral by just looking at the graph.

   (b) The coefficients of the cosine terms lead us to a piecewise integral:

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nx)\, dx + \int_{0}^{\pi} \cos(nx)\, dx\right)$$

   Evaluate this integral.

   (c) Write an integral for b_n. Evaluate this integral to find the coefficients of the sine terms in the Fourier series expansion.

   (d) Some of the coefficients you calculated in Parts (a)–(c) should have come out equal to 0. Explain how you could have predicted some of those without doing any calculations, just by looking at the plot of the square wave.

   (e) Write the Fourier expansion for this square wave.

2. *[This problem depends on Problem 1.]* For the series you found in Problem 1, have a computer draw the 1^{st}, 5^{th}, 20^{th}, and 100^{th} partial sums. Sketch the results and verify that they are approaching the original square wave. What happens at the x-values for which the original function is discontinuous?

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3. Plot each function $g(x)$ given below on the same plot with the function $f(x) = 13\cos(2x)$. For each one describe how the plot of $g(x)$ is similar to or different from the plot of $f(x)$.

   (a) $g(x) = 5\cos(2x) + 12\sin(2x)$

   (b) $g(x) = 12\cos(2x) + 5\sin(2x)$

   (c) $g(x) = 10\cos(2x) + 24\sin(2x)$

   (d) $g(x) = 5\cos(4x) + 12\sin(4x)$

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4. The function $f(x)$ equals x from $x = -4$ to $x = 4$ and is extended periodically from there, with a period of 8.

   (a) How can you know without doing any calculations that the Fourier series for $f(x)$ contains only sine terms?

   (b) Write a sine function with period 8.

   (c) Write a sine function with period 4. This function will also repeat when x increases by 8.

   (d) What is the next largest period that a sine function can have and still repeat when x increases by 8? Write a sine function with that period.

   (e) Write a Fourier series for $f(x)$ without the coefficients filled in: $f(x) = \sum b_n \sin(\text{something } x)$ where b_n is not-yet-determined, but where you do fill in what that "something" is.

   (f) Find the coefficients b_n and write the Fourier series for $f(x)$. Simplify your answer as much as possible.

5. *[This problem depends on Problem 4.]* For the series you found in Problem 4, have a computer draw the 1^{st}, 5^{th}, 20^{th}, and 100^{th} partial sums on the domain $[-12, 12]$. Sketch the results and explain why they make sense.

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6. In this problem you'll find a Fourier series of the function $f(x) = e^x$ on the domain $0 \le x \le 5$.

   (a) We define a new function $g(x)$ that is an odd extension of $f(x)$. In other words, in the domain $0 \le x \le 5$, $g(x) = e^x$. In the domain $-5 < x \le 0$, $g(x) = -e^{-x}$. Beyond that $g(x)$ repeats with period 10. Sketch a graph of $g(x)$ from $x = -30$ to $x = 30$.

   (b) If you create a Fourier series to represent $g(x)$, you know without doing any work that either the a_n or the b_n coefficients will be zero. Which one, and how do you know?

   (c) Write down integrals for the non-zero coefficients. Note that you will have to split your integrals into two parts: one going from -5 to 0, and the other from 0 to 5.

   (d) Explain how you can know without doing any calculations that both parts of your integral will give the same result. This means you can simply evaluate one of them and multiply the result by 2.

   (e) Evaluate your integrals and write the resulting Fourier series. Simplify as much as possible.

7. *[This problem depends on Problem 6.]* For the series you found in Problem 6, have a computer draw the 1^{st}, 5^{th}, 20^{th}, and 100^{th} partial sums on the domain $[-10, 10]$. Sketch the results and explain why they make sense.

For Problems 8–10, find a Fourier sine series *and* a Fourier cosine series for the given function on the given domain.

8. $f(x) = x^2$ on $0 \le x \le 2$

9. $f(x) = x^4$ on $0 \le x \le 2$

10. $f(x) = \sin x$ on $0 \le x \le \pi$

For Problems 11–14 calculate the Fourier transform of $f(x)$. Set up the integral by hand, but you may use a computer to evaluate it for you. Then sketch $f(x)$ and $\hat{f}(p)$.

11. $f(x) = e^{-3x^2}$

12. $f(x) = 5e^{-3x^2}$

13. $f(x) = 1/(1 + x^2)$

14. $f(x) = (\cos x)e^{-x^2}$

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15. In this problem you'll see how a Fourier series becomes a Fourier transform in the correct limit.

    (a) Let $f(x) = e^{-x^2}$ from $x = -1$ to 1, repeated periodically thereafter. Find the complex exponential Fourier series for $f(x)$ and plot the coefficients as a function of frequency p (not of n), with p going from -10 to 10.

    (b) Repeat Part (a) for $f(x) = e^{-x^2}$ from $x = -2$ to 2, repeated periodically thereafter.

    (c) Repeat Part (a) for $f(x) = e^{-x^2}$ from $x = -10$ to 10, repeated periodically thereafter.

    (d) What is happening to the frequencies of the coefficients as you increase the period? What is happening to the shape of the plot? What is happening to the height of the plot?

    (e) Plot the Fourier transform of $f(x) = e^{-x^2}$ as p goes from -10 to 10. How does this plot compare to the previous ones in what frequencies are included, the shape of the plot, and the height of the plot?

16. Create an animation showing the plot of the Fourier transform of $f(x) = Ae^{-(x/x0)^2}$ as you vary A and $x0$ over some ranges of values. Describe the effect each of them has on the plot of $\hat{f}(p)$.

17. Consider the points $f(0) = 2$, $f(3) = 7$, $f(6) = -2$, $f(9) = 1$, $f(12) = 4$, $f(15) = -3$, $f(18) = 2$, $f(21) = 2$.

    (a) Calculate the DFT coefficients $\hat{f}_n$ for $0 \le n \le 7$.

    (b) On one plot show the real and imaginary parts of $(1/8)\sum_{n=0}^{7} \hat{f}_n e^{2\pi inx/12}$. Show the original 8 points f_r on the same plot.

    (c) Repeat Part (b) for $-3 \le n \le 4$. *Hint*: you can easily get all the $\hat{f}_n$ values you need from the ones you already calculated.

    (d) On one plot show the original points f_r and the real and imaginary parts of $(1/8)[(1/2)(\hat{f}_{-4}e^{-8\pi inx/12} + \hat{f}_4 e^{8\pi inx/12} + \sum_{n=-3}^{3} \hat{f}_n e^{2\pi inx/12}]$.

    (e) You should have found that all three functions you just defined go through all of the original data points f_r, but behave differently in between them. Looking at your plots, which of these functions is likely to be the most useful for modeling your original function, and why?

18. Have a computer generate a table of 256 points $f(x)$ where x goes from 0 to 30. At each point let $f(x) = 2\sin(2x) + 2\sin(5x) + R$, where R is a random number (different at each point) from -4 to 4. Plot the data points on a graph of $f(x)$ vs. x. Then take a discrete Fourier transform of the data and plot the results on a graph of $\hat{f}(p)$ vs. p. Explain why the results appear the way they do.

19. Consider the function $f(x) = e^{\cos x}$.

(a) Have a computer generate a table of 200 values of $f(x)$ ranging from $x = 0$ to $x = 99.5$. Plot these values on a graph of f vs. x. (Just plot the values in your table, not the curve for the function.)

(b) Take a discrete Fourier transform of your list of values. What frequency is associated with the n^{th} term in that list? Plot the moduli of the values of the discrete Fourier transform from $n = 1$ to $n = 100$, with the horizontal axis showing the associated frequencies. (Leave out $n = 0$, which just represents the constant term.)

(c) What frequency shows the highest peak in the DFT? What does that tell you about the original function? *Hint*: you could have predicted this peak before doing the calculations.

(d) What frequency shows the second highest peak in the DFT? What does that tell you about the original function? *You probably couldn't have predicted this one.*

20. Consider the list $f_r = (1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 3, 2)$.

(a) Plot these data points, using the x-values $1, 2, 3, 4, 5, 6, \ldots$.

(b) Take a discrete Fourier transform of the list, then take an inverse discrete Fourier transform, and then plot the resulting points. This plot should look identical to the previous one.

(c) Take a discrete Fourier transform of f_r, set $\hat{f}_0 = 0$ in the resulting list, then take its inverse discrete Fourier transform and plot the results. How does this look different from your original plot?

(d) Take a discrete Fourier transform of f_r, set $\hat{f}_2 = \hat{f}_{-2} = 0$, take an inverse discrete Fourier transform, and plot the results. How does this plot look different from your original plot? *Hint*: some computers store the results in the order $\hat{f}_0$–$\hat{f}_{N-1}$, so the component $\hat{f}_{-2}$ may be stored as $\hat{f}_{N-2}$.

21. Choose an image, import it into a computer algebra program, take a discrete Fourier transform of the image data, set the high frequency components to zero, and inverse transform it to get a new image. Describe how the new image looks similar to, and different from, the original. You may need to do some trial and error to figure out exactly which components to eliminate. Be aware that when you eliminate Fourier coefficients the inverse discrete Fourier transform will usually give you complex numbers. The easiest way to deal with this is to just take the real part after you take the IDFT. *Hint*: If you use a color image then each pixel will most likely be a list of three numbers instead of a single number. You can still do the same thing. Each component in Fourier space will be a list of three numbers and you can set them all equal to zero for the high frequency modes.

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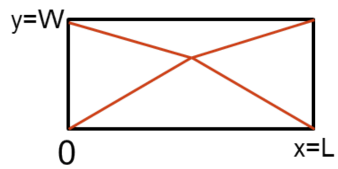
22. Consider a rectangular drumhead pulled up into four planes.



   (a) Assuming the drumhead extends along $0 \le x \le L$ and $0 \le y \le W$, and the middle point is raised to height H above the plane, write the equations for the four planes.

   (b) To create a Fourier series for this shape, you need to create a periodic extension. Is it best to use an odd or even extension in x? In y? Explain.

   (c) Write the Fourier series for your periodic function and write the integrals for finding the coefficients. You do *not* need to evaluate those integrals for this problem.

23. *[This problem depends on Problem 22.]*

   (a) Evaluate the integrals you found in Problem 22 for the Fourier coefficients. (You could do this by hand but it would be very tedious. Whether you do it by hand or on a computer, you will need to consider the case $m = n$ separately from the case $m \neq n$.)

   (b) Set $L = 4$, $W = 2$, and $H = 1$ and plot the Fourier series, including all terms up to $m = n = 20$. Verify that it matches the original shape of the drumhead.

For Problems 24–25, find the Fourier series of the indicated function twice, first using sines and cosines and then using complex exponentials. If it is given over a finite domain assume that it extends periodically beyond that domain. (Do not assume an even or odd extension.)

24. $f(x) = \sin^5 x$. *Hint*: don't just naively accept what the computer gives you if it doesn't make sense.

25. $f(x) = x^2 \sin x$, $0 < x < \pi$

26.

   (a) Write a Fourier series that represents the function $y = x^2$ from $x = -20$ to $x = 20$.

   (b) Draw the 1^{st}, 5^{th}, 20^{th}, and 100^{th} partial sums of the resulting series on one plot, along with $y = x^2$.

27. Let $f(x) = x^{-1/3}$ from $-\pi$ to π and repeat thereafter.

   (a) This function only satisfies the Dirichlet conditions if $\int_{-\pi}^{\pi} f(x)dx$ is finite. Show that it is. (Because of the vertical asymptote, you have to break it up into $\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$ and show that both parts are finite.)

   (b) Use a computer to generate the 20^{th} partial sum of the Fourier series for this function between $x = -\pi$ and $x = \pi$, and then plot that series from $x = -2\pi$ to $x = 2\pi$.

   (c) The resulting graph should look a lot like $x^{-1/3}$ between $-\pi$ and π. What does the graph do at $x = 0$ and why? What does the graph do at $x = \pi$ and why?

   (d) What are the values of the Fourier series at $x = 0$ and $x = \pi$? Why do these values make sense?

28. **The Gibbs Phenomenon** If a function $f(x)$ satisfies the Dirichlet conditions then its Fourier series converges to its value where it is continuous and converges to the average of its left and right limits at jump discontinuities. However, the convergence at such discontinuities suffers from a problem known as the Gibbs phenomenon. You'll explore this by considering the function $f(x)$ defined as the odd extension of $f(x) = 1$ from $x = 0$ to $x = 1$.

    (a) Find the Fourier sine-cosine series of $f(x)$. (It should only have sines.)

    (b) Plot the partial sum of the Fourier series including the first three nonzero terms. Your plot should go from $x = -1$ to $x = 1$ and should include horizontal lines at $y = -1$ and $y = 1$ for reference.

    (c) For negative x the series should be near -1 and for positive x its should be near 1, but in going from negative to positive it "overshoots" by a bit, going higher than 1. Looking at your plot, estimate the amount it overshoots by.

    (d) Show all of the partial sums up through the sixth nonzero term together on one plot. As you use more terms, you should find that the amount by which the series overshoots 1 does *not* decrease, but that it moves back down near 1 more quickly.

    (e) Plot the 50^{th} partial sum of the Fourier series and show that it still overshoots by the same amount. Zoom in your plot enough to estimate the value of x at which it returns back to 1 after overshooting.

    (f) The fact that a Fourier series overshoots a jump discontinuity by an amount that doesn't decrease as you add more terms is the Gibbs phenomenon. Explain how we can still say that the series converges to the function even though this occurs.