Motivating Exercise for Complex Analysis: Flow Around a Rock

Figure 1.1 shows a straight river with a rock jutting out from one bank. The lines in the picture represent "streamlines" of the river: if you drop a small splinter of wood into the water, it will flow along a streamline.

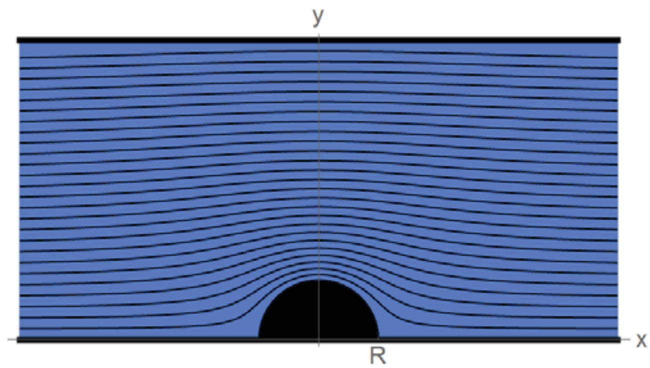
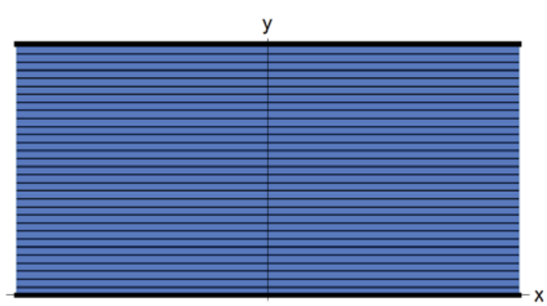


Figure 1.1: Stream with rock and streamlines

Our goal is to mathematically describe the water flow. The most general description would be the vector function $\vec{v}(x, y)$ that represents the water's velocity at every point. But under certain circumstances[1] you can start by finding the *scalar* function $\psi(x, y)$ called its "stream function." There are three things you need to know about this function.

- The curves of constant ψ are the streamlines of the flow (the lines that bend around the rock in Figure 1.1).
- Once you have the scalar ψ, it's easy to find the velocity $\vec{v}$ that completely describes the water's motion.[2]
- The stream function satisfies Laplace's equation $d^2\psi/dx^2 + d^2\psi/dy^2 = 0$. If you can find a function that obeys that equation and correctly lines up along your boundaries (along the banks and around the rock in our example), you have found your stream function and solved the problem.

Let's start with a simpler case: a horizontal stream with straight banks and no obstacles at all.



[1]Throughout this exercise we consider laminar, irrotational flow of an incompressible fluid. You can still define a stream function if the flow is not irrotational but it doesn't obey the third condition listed above. If you have no idea what these terms mean don't worry about it: we put them in a footnote to assure you that you can do the exercise without them.

[2]Velocity is $\vec{v} = (d\psi/dy)\hat{\imath} - (d\psi/dx)\hat{\jmath}$ if you care, but our point here is that finding ψ is the goal.

1. Explain how we know that the streamlines, which in general are curves, will in this case be horizontal lines. (Only a sentence or two is required here, but "they look horizontal in the picture" is not sufficient.)

2. The streamlines are the curves along which $\psi(x, y)$ is constant. Explain why any function $\psi(y)$ (no x-dependence) will give us the horizontal streamlines we want.

3. Explain why a linear function $\psi(y) = ky + C$ is the only function of y that can satisfy Laplace's equation.

That's it! You have found your first stream function, and mathematically solved the problem of a horizontal stream with no obstacles. In general we ignore the $+C$ (it makes no physical difference) and write $\psi = ky$.

Now let's go back to Figure 1.1, putting a half-disk of radius R at the edge of the river. We will choose to put the origin of our coordinate system at the middle of this disk. The stream function for that more complicated river must meet four criteria.

- There must be a streamline along the bottom of the stream; in other words, the stream function must be constant along $y = 0$.
- There must be a streamline going along the edge of the rock; in other words, the stream function must be constant along $y = \sqrt{R^2 - x^2}$.
- Far away from the rock, the rock must make little difference; in other words, for y much larger than R, the stream function should approach the $\psi = ky$ we calculated above.
- Finally, the stream function must satisfy Laplace's equation.

Can you think of such a function? That's easy to answer: no you can't! So we're going to give you one.

$$\psi(x, y) = ky\left(1 - \frac{R^2}{x^2 + y^2}\right)$$

The curves plotted in Figure 1.1 are the lines of constant ψ for this function.

4. Verify that this stream function meets all four criteria we outlined for it.

You should have been able to convince yourself of the following facts about this stream function.

- It satisfies the boundary conditions and Laplace's equation. In other words it is a valid solution to the mathematical problem we asked you to solve.
- The contour lines of ψ look like plausible streamlines for laminar flow of a fluid around a semicircular obstacle.
- It's not at all obvious how we came up with it.

Problems like this one come up commonly in fluid dynamics. More generally, the problem "find the function that satisfies Laplace's equation subject to those boundary conditions" comes up in many areas of engineering and physics, including electromagnetism and thermodynamics. Such problems and their solutions involve only real numbers, but the easiest way to solve them is often to use complex analysis. When you have learned more about complex functions and their derivatives, you will be able to figure out stream functions like the one that we mysteriously dropped on you in this exercise.