Discovery Exercise for Euler's Formula I—The Complex Exponential Function

Some of the simplest oscillatory functions result from the following differential equation:

$$\frac{d^2y}{dx^2} = -y \tag{1}$$

1. Show that $y = A\cos x + B\sin x$ is a valid solution to Equation 1 for any values of the constants A and B.

2. Explain how we know that $y = A\cos x + B\sin x$ is the *general* solution to Equation 1.

3. Show that $y = e^{ix}$ is a valid solution to Equation 1.

Since $y = A\cos x + B\sin x$ is the general solution to Equation 1, any other solution can be rewritten in that form. So we must be able to write:

$$e^{ix} = A\cos x + B\sin x \tag{2}$$

It is vital to note that Equation 2 is not true for *all* values of A and B. e^{ix} represents a particular solution to Equation 1, which means that Equation 2 must be true for *some particular* values of A and B; our job is now to find them.

4. Plug $x = 0$ into both sides of Equation 2 to find one of the two constants.

5. Take the derivative with respect to x of both sides of Equation 2 and then plug in $x = 0$ to find the other constant.

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6. The function e^{kt} is used to describe a growing quantity when k is a positive real number, and a decaying quantity when k is a negative real number. Based on the results you've found in this exercise, what kind of quantity does e^{kt} describe when k is imaginary?