# Computer Problems for Complex Numbers

*The two problems below are a set; the first should be done without a computer and the second is a computer-based follow up.*

1. An underwater mass on a spring experiences two forces: $F_{spring} = -kx$ and $F_{water} = -cv$. Newton's Second Law for this system is $ma = -cv - kx$. For this problem, assume the mass $m = 2$ kg and the spring constant $k = 5$ N/m. Higher values of the constant $c$ indicate a stronger damping force.

   (a) If $c = 6$ kg/s the system is "underdamped." Solve the differential equation in this case and show that the resulting motion is oscillatory.

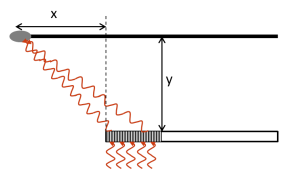
   (b) If $c = 7$ kg/s the system is "overdamped." Solve the differential equation in this case and show that the resulting motion is *not* oscillatory.

   (c) Somewhere between $c = 6$ kg/s and $c = 7$ kg/s is the value known as "critically damped." For any $c$-value below this critical value, the system will oscillate. Find the $c$-value that leads this system to be critically damped.

2. *[This problem depends on Problem 1.]* Have a computer generate three plots on the same set of axes, one for the underdamped case, one for the overdamped case, and one for the critically damped case. You may choose any initial conditions you wish other than starting at rest at the origin, but you should use the same initial conditions for all three cases. Clearly indicate which plot is which.

3. **Exploration: Diffraction Grating**

In a diffraction experiment light passes through an array of $N$ thin slits, lined up in a row with spacing $\delta$ between adjacent slits. The light has wavelength $\lambda$. You may assume the light beams all have zero phase and equal amplitude $A$ as they pass through the slits. The light is detected on a wall parallel to the wall with the slits, a distance $y$ behind it. The detector is placed on the back wall at a distance $x$ beyond the first slit.




(a) How far does the light travel as it moves from the $n^{th}$ slit to the detector?

(b) Use a Maclaurin series to approximate your answer to Part (a) as a linear function in $\delta$. Under what circumstances is this a reasonable approximation to use?

(c) Using the approximate distance you found, what phase change does the light experience as it goes from the $n^{th}$ slit to the detector? (The phase difference can be defined as $2\pi$ times the number of wavelengths between the slit and the detector.)

(d) The signal received from a single slit is $A\cos(\omega t + \phi)$, where $\omega$ is the angular frequency of the light beam and $\phi$ is the phase change from the slit to the detector. Write a sum representing the total signal received at the detector.

(e) The total signal can be rewritten in the form $\sum A\cos(\omega t + c + dn)$. Find $c$ and $d$ in terms of the constants given in the problem.

(f) Rewrite the signal as the real or imaginary part of a complex exponential function. Find the sum of this complex series. Give your answer in terms of $c$ and $d$, which will be much shorter and easier than writing it in terms of the original constants in the problem.

Your sum should have been in the form $Ze^{i\omega t}$, where $Z$ is a complex constant. The total signal received at the detector is the real part of your answer, but without calculating that you can see that the signal will oscillate with an amplitude $|Z|$. Even finding that modulus is a fairly tedious calculation, however, so let's turn to a computer. Let $A = 1$, $y = 9$ m, $\delta = 7 \times 10^{-4}$ m, and $\lambda = 10^{-6}$ m. Our goal is to see the pattern of light and dark that will emerge on the wall, so we need leave $x$ as a variable.

(g) Plug your solution from Part (f) into a computer using $N = 2$ for a double-slit experiment. Ask the computer to plot the amplitude of the signal as a function of $x$. What pattern emerges on the wall?

(h) Repeat Part (g) for 5 slits, 10 slits, and 100 slits. How does the pattern change as the number of slits increases?

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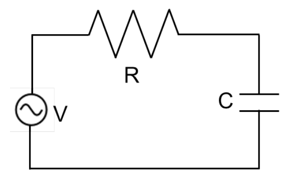
4. Consider the circuit shown below.




If the voltage source produces a voltage like $V = V_0 \sin(\omega t)$ then you know by now how to solve for the resulting current. In this problem, however, you will consider a voltage source $V = V_1 \sin(\omega_1 t) + V_2 \sin(\omega_2 t)$, and you'll solve for the current using the principle of linear superposition.

(a) Find the complex impedance of this circuit. (The answer will depend on $\omega$, the frequency of the voltage source. Leave $\omega$ as a variable in this part; you'll fill in specific frequencies in the next parts.)

(b) Using the impedance you found in Part (a), find the complex current $\mathbf{I}$ that would result from a voltage source $V = V_1 \sin(\omega_1 t)$.

(c) Using the impedance you found in Part (a), find the complex current $\mathbf{I}$ that would result from the voltage source $V = V_2 \sin(\omega_2 t)$.

(d) Write the differential equation satisfied by $\mathbf{I}$ and show that the sum of the two currents you found is a solution to this differential equation.

(e) Write your solution $\mathbf{I}(t)$ for $R = 10^5 \Omega$, $C = 10^{-6}$ F, $V_1 = 3$ V, $V_2 = 5$ V, $\omega_1 = 2$ s$^{-1}$, and $\omega_2 = 3$ s$^{-1}$. Plot $V(t)$ and the imaginary part of $\mathbf{I}(t)$ on the same plot. Include a range of times sufficient to see the behavior of the functions.

5. **Exploration: Other Driving Functions** *[This problem depends on Problem 4.]* The circuit from Problem 4, with the impedance you calculated in Part 4(a), is driven by a different voltage source. This source produces a "square wave": $V = 1$ for $0 \le t < 1$, then $V = -1$ for $1 \le t < 2$, and this pattern repeats indefinitely with period 2. Such a function can be represented as a sum of an *infinite number* of sine waves, a "Fourier sine series." Finding the coefficients is a topic for another chapter, so here we just give them to you:

$$V(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin(n\pi t) \text{ (odd } n \text{ only)} = \frac{4}{\pi} \sin(\pi t) + \frac{4}{3\pi} \sin(3\pi t) + \frac{4}{5\pi} \sin(5\pi t) + \ldots$$

(As in Problem 4, just leave the resistance and capacitance as $R$ and $C$ until we tell you to put in numbers. Leaving them as letters for now will keep your equations more readable.)

(a) Plot the driving function $V(t)$ from $t = 0$ to $t = 6$. On the same plot show the first term of the series expansion: $(4/\pi)\sin(\pi t)$.

(b) Make two more plots, each showing the function $V(t)$ and a partial sum of its series expansion. On the first one put the sum through $n = 5$. On the second one put the sum through $n = 101$. Describe what happens to the partial sum as you include more terms.

(c) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the first term in its series expansion (the $n = 1$ part of the series).

(d) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the sum of the first two nonzero terms in its series expansion.

(e) Write the complex solution $\mathbf{I}(t)$ that you get if you replace the voltage $V(t)$ with the $\sin(n\pi t)$ term *(not the partial sum up to that term)* of its series expansion.

(f) Write an infinite series for the solution $\mathbf{I}(t)$ that you get using the complete infinite series expansion for $V(t)$.

(g) Now let $R = 10^5\Omega$ and $C = 10^{-6}$ F. Plot the partial sums of $Im(\mathbf{I}(t))$ for $n = 1$, $n = 11$, $n = 21$, $n = 31$, and $n = 101$, five plots in all. (Remember to plot partial sums, not individual terms.) Each of your plots should go from $t = 0$ to $t = 4$.

(h) Describe the behavior of the $n = 101$ plot. What does the current do at $t = 0$? What does it do shortly after that? What does it do at $t = 1$, and shortly after that? Describe what the capacitor is doing physically to produce the behavior you see. (If you skipped the computer part you can still do this part by *predicting* the behavior of $I(t)$ based on what you would expect the capacitor to do.)