Discovery Exercise: Variational Problems and the Euler-Lagrange Equation

As you work through this exercise your first question may well be "why on Earth would anyone want to do this?" We always encourage that question, but put it on hold for the time being. By the end of the chapter we hope to have convinced you that problems like this one can hold great importance.

Here is the problem. In each of Parts 1–3 we are going to specify a curve $y(x)$ that extends from $(0,0)$ to $(1,1)$. You are going to compute the following quantity along each curve.

$$\int_0^1 \left(y'^2 + 10xy\right) dx$$

Example problem: $y = x^2$

Solution to example problem: We replace y with x^2, and therefore y' with $2x$, in the formula we are integrating.

$$\int_0^1 \left[(2x)^2 + 10x(x^2)\right] dx = \int_0^1 \left(4x^2 + 10x^3\right) dx = \frac{4}{3}(1)^3 + \frac{5}{2}(1)^4 - 0 \approx 3.83$$

1. $y = x$

2. $y = x^3$

3. $y = x^4$

4. Draw a pretty large graph with $(0,0)$ at the bottom left and $(1,1)$ at the top right. On this graph draw the four curves above and label each curve with a number representing its integral. For instance, the curve $y = x^2$ should be labeled with the number 3.83.

5. A "variational problem" calls for you to find the curve that *minimizes* an integral such as this one. Based on your results, sketch in the curve that you think would minimize this particular integral.