CHAPTER 14

Calculus of Variations

Before you read this chapter, you should be able to…

- solve ordinary differential equations (ODEs) using the methods of "separation of variables" and "guess and check" (Chapter 1).

It is also helpful, but not essential, to be able to solve basic optimization problems. This skill in one variable is not discussed in this book, although the multivariate equivalent is discussed in Chapter 4. In the same helpful-but-not-essential category come line integrals (Chapter 5).

After you read this chapter, you should be able to…

- explain what a "variational problem" is.
- use the "Euler-Lagrange equation" to solve variational problems.
- derive the Euler-Lagrange equation.
- solve mechanics problems by using the "principle of least action" to represent them as variational problems.

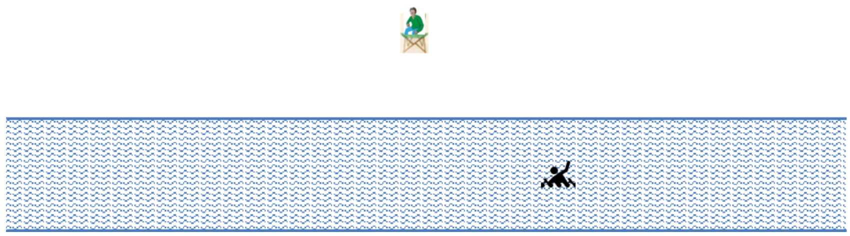
An optimization problem seeks to minimize or maximize some value. In first year calculus you find one number (such as a position or time) that optimizes your objective. In multivariate calculus you might find two or more numbers, such as a point (x, y) on a plane. In a "variational problem" your goal is to find an entire function. An example would be finding the shortest path between two points along a curved surface like a cone.

In Section 14.2 you will learn the "Euler-Lagrange equation." This remarkable formula starts with a variational problem and produces a differential equation; if you can solve that equation, you have found the function that optimizes your objective. In Section 14.3 you will see where the Euler-Lagrange equation comes from. Section 14.4 will highlight one particularly important application of this technique, the Lagrangian formulation of classical mechanics.

14.1 Motivating Exercise: Rescuing the Swimmer

Each chapter begins with a "motivating exercise" for the students to work through in class or as homework before studying the chapter. They are optional, but if you use them it can help the students understand why they are learning the material in the chapter.

You are working as a river lifeguard when you see a man starting to drown.



In each of the scenarios below, the question is the same: draw the path that will take you most quickly to the drowning man. In the first scenario you can draw the path exactly. In the others, your goal is a rough qualitative sketch.

- You move at the same speed in water as you do on land.
- Your speed on land is twice as fast as your speed through the water.
- The water gets deeper as you get farther from shore, and you are walking the whole way. So you enter the water at the same speed you had on land, but the farther you get from shore, the slower you walk.

We wish to call your attention to several features of this problem. It is an optimization problem, in which you are trying to minimize a particular quantity (time). But especially in the third scenario, you are not minimizing time by finding some other numerical variable (as in traditional optimization problems); you are finding a *function* (a curve) that minimizes the time. Problem of the form "find the function that minimizes this integral" will occupy this entire chapter.

14.2 Variational Problems and the Euler-Lagrange Equation

A "variational problem" means "find the curve that minimizes this integral." The Euler-Lagrange equation replaces such a problem with a differential equation. When you solve the differential equation (using for instance the techniques of Chapters 1 and 10), you find the function that minimizes the integral.

14.2.1 Discovery Exercise: Variational Problems and the Euler-Lagrange Equation

Most sections begin with a "discovery exercise" that students can do in class or as homework before covering the section. They are optional, but if the students do them they will derive some of the key math ideas themselves.

As you work through this exercise your first question may well be "why on Earth would anyone want to do this?" We always encourage that question, but put it on hold for the time being. By the end of the chapter we hope to have convinced you that problems like this one can hold great importance.

Here is the problem. In each of Parts 1–3 we are going to specify a curve $y(x)$ that extends from $(0, 0)$ to $(1, 1)$. You are going to compute the following quantity along each curve.

$$\int_0^1 \left(y'^2 + 10xy\right) dx$$

Example problem: $y = x^2$
Solution to example problem: We replace y with x^2, and therefore y' with $2x$, in the formula we are integrating.

$$\int_0^1 \left[(2x)^2 + 10x(x^2)\right] dx = \int_0^1 \left(4x^2 + 10x^3\right) dx = \frac{4}{3}(1)^3 + \frac{5}{2}(1)^4 - 0 \approx 3.83$$

1. $y = x$
2. $y = x^3$
3. $y = x^4$
4. Draw a pretty large graph with $(0, 0)$ at the bottom left and $(1, 1)$ at the top right. On this graph draw the four curves above and label each curve with a number representing its integral. For instance, the curve $y = x^2$ should be labeled with the number 3.83.
5. A "variational problem" calls for you to find the curve that *minimizes* an integral such as this one. Based on your results, sketch in the curve that you think would minimize this particular integral.

In Problem 14.3 you will return to this function and see how close your sketch came to the optimal curve.

14.2.2 Explanation: Variational Problems and the Euler-Lagrange Equation

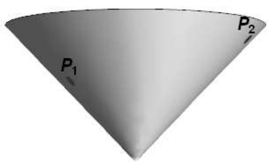
The following are all examples of variational problems.

1. Find the shortest path between two points on a plane. On a sphere. On a cone.
2. You're going to roll a toy car made of pinewood down a curved track that ends 10 feet down and 40 feet across from where it started. What track shape will get the car to the finish line in the least amount of time?
3. Draw a curve between the origin and the point $(1, 1)$ and then revolve that curve around the y axis to form a surface of revolution. What curve will minimize the surface area of the resulting surface? (The solution to this problem describes the shape of a soap bubble with appropriate constraints.)

In each case you are trying to *find the function* that *minimizes an integral.*[1] Below we focus on one of these examples; you will solve the rest in the problems.

Skating on an Ice Cream Cone

The picture below shows the cone $z^2 = x^2 + y^2$ (for $z \geq 0$), and two points on that cone. What is the shortest path along the cone from point P_1 to point P_2?



[1]Most variational problems involve minimization, but you could be maximizing instead.

The most obvious approach is to approximate a straight line as closely as possible, always moving around-and-up. But circling around the cone takes less distance at lower heights, so perhaps we should go around horizontally from P_1 and then up to P_2. Or maybe even dip down a little below P_1 as we go around. It isn't obvious, is it?

To approach this problem quantitatively we need a formula for the total distance along any given curve. Then we will need to find the curve that minimizes that formula.

A natural way to specify any given point on this cone is by giving its cylindrical coordinates z and ϕ. An arbitrary small step along this cone changes both of those variables, and the distance of such a step is $ds = \sqrt{2\,dz^2 + z^2 d\phi^2}$. That formula is *not* obvious, and you will derive it two different ways: geometrically in Problem 14.17 and algebraically in Problem 14.18. But here we want to focus on what to do with that formula once we have it.

As a first step we need to express the total distance along a curve in terms of one variable. You cannot express an arbitrary location on the cone with one variable, but you can express an arbitrary location *on any given curve* with one variable. If a curve is defined by a function $\phi(z)$, we can use that function to replace $d\phi$ with dz.

$$s = \int ds = \int \sqrt{2\,dz^2 + z^2 d\phi^2} = \int \sqrt{2\,dz^2 + z^2 \left(\frac{d\phi}{dz}dz\right)^2} = \int_{z_A}^{z_B} \sqrt{2 + z^2 \phi'^2}\,dz \qquad (14.2.1)$$

Focus on the middle step in that sequence: the replacement of $d\phi$ with $(d\phi/dz)dz$ to set up an integral in one variable represents a fairly common technique in such problems.

At this point, if we gave you a particular curve—that is, a function $\phi(z)$ and the coordinates of two points P_1 and P_2—you would know just what to integrate to find the total distance. That would be the sort of problem we asked you in the Discovery Exercise (Section 14.2.1).

But our job is to find the $\phi(z)$ curve that *minimizes* the integral in Equation 14.2.1. First we're going to talk briefly about variational problems in general, and then we will introduce the formula that solves them. Finally, with new tools in hand, we will circle back to our cone problem.

The Generic Variational Problem

Any variational problem can be expressed in the following way.

1. You are given two points (x_o, y_o) and (x_f, y_f). You are going to find a curve, expressed as $y(x)$, that extends from the first point to the second. There are of course infinitely many such curves.
2. You are also given a function $f(x, y, y')$. At any given point on any given curve, x and y and y' have specific values so f has a specific value. Note, however, that the *same* point on a *different* curve might have a different y' and therefore a different value of f.
3. Your job is to find the curve that minimizes the following integral:

$$\int_{x_o}^{x_f} f(x, y, y')\,dx \qquad \textit{the generic objective function} \qquad (14.2.2)$$

That may look hopelessly abstract, but it's often easy to calculate for specific examples. For instance, in the Discovery Exercise (Section 14.2.1) the function is $f(x, y, y') = y'^2 + 10xy$ and the endpoints are $(x_o, y_o) = (0, 0)$ and (x_f, y_f)=(1,1). To integrate this function along the curve $y = x^2$, we replace y with x^2 in the integral and y' with $2x$.

$$\int_0^1 \left[(2x)^2 + 10x(x^2)\right] dx = \int_0^1 \left(4x^2 + 10x^3\right) dx = \frac{4}{3}(1)^3 + \frac{5}{2}(1)^4 - 0 \approx 3.83$$

In the Discovery Exercise you integrate the same function along a few other curves. We urge you to give that a try if you haven't already; five minutes of setting up integrals will do you more good than twenty minutes of staring at the last few paragraphs.

You may also want to look back at the cone example above. Note how a geometrical scenario led us to minimizing $\int_{z_A}^{z_B} \sqrt{2+z^2\phi'^2}\,dz$, and make sure you see how this fits the template of a variational problem.

Finally, it can be instructive to consider how this topic relates to line integrals. In a variational problem we are given a particular function to integrate along a particular curve. Isn't that exactly what a line integral does? Well, yes and no.

A variational problem starts with a function $f(x, y, y')$ and a curve. At each step along the curve it multiplies the value of the function by the horizontal distance dx, and then adds them up. That's what Equation 14.2.2 says.

A line integral[2] starts with a function $f(x, y)$ and a curve. At each step of the curve it multiplies the value of the function $f(x, y)$ by the diagonal distance ds, which can be expressed as $\sqrt{1+y'^2}\,dx$, and then adds them up.

$$\int_{x_o}^{x_f} f(x,y)\,ds = \int_{x_o}^{x_f} f(x,y)\sqrt{1+y'^2}\,dx \qquad \textit{the generic line integral} \qquad (14.2.3)$$

Our point, and we hope you can see it, is that Equation 14.2.3 is a special case of Equation 14.2.2. Every line integral can be used as the basis for a variational problem, but not every variational problem comes from optimizing a line integral.

The Euler-Lagrange Equation

If you've followed us to this point you understand what kinds of functions we are integrating, and why we might want to minimize those integrals. But we haven't said anything yet about *how* to minimize them.

We're ready now to jump to the answer. In an introductory calculus optimization problem you find the "critical points" where $f'(x) = 0$ and you know that your minimum or maximum must be at one of them. In a variational problem you find the "stationary solutions" by plugging into the formula below, and your solution will be one of them. Usually the boundary conditions will restrict you to one stationary solution and you're done.

The Euler-Lagrange Equation

The function $y(x)$ that minimizes or maximizes the integral

$$\int_{x_0}^{x_f} f(y, y', x)\,dx$$

subject to the boundary conditions $y(x_0) = y_0$, $y(x_f) = y_f$ obeys the following differential equation, subject to the same boundary conditions.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \qquad (14.2.4)$$

We will derive Equation 14.2.4 in Section 14.3. Here we want to focus on how and when to use it, starting with this cautionary note: mind the distinction between partial and total derivatives. When you evaluate $\partial f/\partial y'$ you treat y' as the variable and everything else as a constant, so the derivative of y or x is zero. Similarly when you evaluate $\partial f/\partial y$. But the d/dx operator is a total derivative, so the derivative of y is y' and the derivative of y' is y''.

[2] To be precise, "the line integral of a scalar function in two dimensions"

EXAMPLE **The Euler-Lagrange equation**

Problem:
Find the function $x(t)$ that minimizes the integral $\int \left[(x+\dot{x})^2 + 3x^2\right] dt$ from $x(0) = 0$ to $x(1) = 1$. (Remember that $\dot{x}$ means the derivative of x with respect to time.)

Solution:
First calculate the derivatives in the Euler-Lagrange equation. Note that t is the independent variable (the role x had in Equation 14.2.4) and x is the function we are varying, so the Euler-Lagrange equation reads $(d/dt)(\partial f/\partial \dot{x}) - (\partial f/\partial x) = 0$. The function f is the integrand, $(x+\dot{x})^2 + 3x^2$.

$$\frac{\partial f}{\partial \dot{x}} = 2(x+\dot{x})$$
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 2(\dot{x}+\ddot{x})$$
$$\frac{\partial f}{\partial x} = 2(x+\dot{x}) + 6x = 2\dot{x} + 8x$$

Plugging these into the Euler-Lagrange equation and dividing by 2 gives $\ddot{x} - 4x = 0$, with solution $x(t) = Ae^{2t} + Be^{-2t}$. To find the arbitrary constants plug in the boundary conditions $x(0) = 0$ and $x(1) = 1$. That gives $A + B = 0$ and $Ae^2 + B/e^2 = 1$ with solution $A = 1/(e^2 - e^{-2})$, $B = -A$.

$$x(t) = \frac{e^{2t} - e^{-2t}}{e^2 - e^{-2}} \quad \text{or} \quad x(t) = \frac{\sinh 2t}{\sinh 2}$$

We'll leave it to you to argue that this must be a minimum rather than a maximum.

A Shortcut for Functions That Involve y' But Not y

Suppose you want to optimize the integral $\int (x/y')dx$.

$$f = \frac{x}{y'} \quad \to \quad \frac{\partial f}{\partial y'} = -\frac{x}{y'^2}, \quad \frac{\partial f}{\partial y} = 0$$

So the Euler-Lagrange equation promises that any stationary solutions must fit the following differential equation.

$$\frac{d}{dx}\left(-\frac{x}{y'^2}\right) = 0$$

If you take that derivative using the quotient rule you get $(-y'^2 + 2xy'y'')/y'^4 = 0$ and brew another cup of espresso to get you through a long night. But there is a shortcut. If d/dx<*something*> is zero, then the <*something*> must be a constant. That leads us to a first order differential equation that we can easily solve.

$$-\frac{x}{y'^2} = C \quad \to \quad y' = C\sqrt{x} \quad \to \quad y = Ax^{3/2} + B$$

(We defined $A = 2C/3$ to simplify the final answer.) Take a moment to convince yourself that you can use this shortcut any time the objective function has no explicit y-dependence.

There is also a shortcut for simplifying problems that involve y and y' but not x. See Problem 14.60.

And Now, Back to Our Cone

Equation 14.2.1 gives the distance along any curve $\phi(z)$ on the cone $z^2 = x^2 + y^2$.

$$s = \int_{z_A}^{z_B} \sqrt{2 + z^2\phi'^2}\, dz$$

Our goal is to find the curve that minimizes this integral. Recognizing a variational problem, we begin by finding a few derivatives.

$$f = \sqrt{2 + z^2\phi'^2} \quad \rightarrow \quad \frac{\partial f}{\partial \phi'} = \frac{z^2\phi'}{\sqrt{2 + z^2\phi'^2}}, \quad \frac{\partial f}{\partial \phi} = 0$$

Did you wonder, when we first wrote Equation 14.2.1, why we represented our curve as $\phi(z)$ instead of $z(\phi)$? The reason was that our integrand, which in its original form was $\sqrt{2\,dz^2 + z^2 d\phi^2}$, explicitly contained a z but no ϕ. When solving for a $\phi(z)$ function with no ϕ in the integrand we can use the shortcut described above: rather than solving $(d/dx)(\partial f/\partial \phi') = 0$ directly, we set $\partial f/\partial \phi'$ equal to a constant C. With a bit of algebra we can solve for ϕ'.

$$\phi' = \frac{\sqrt{2}C}{z\sqrt{z^2 - C^2}}$$

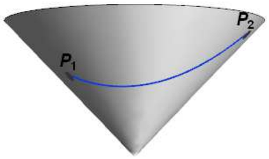
This looks ugly but with the substitution $C = z \sin \alpha$ (or a computer) you can get pretty quickly to the solution.

$$\phi(z) = \sqrt{2}\, \sin^{-1}\left(\frac{C}{z}\right) + D$$

Having solved for a $\phi(z)$ function to get the solution easily, it's easier to work with if we now invert it.

$$z(\phi) = \frac{A}{\sin\left(\phi/\sqrt{2} + B\right)}$$

The constants A and B can be chosen to connect any two arbitrary points on the cone. The picture below shows this curve for two representative endpoints. As we suspected it might, it dips down in the middle for the shortest journey.



14.2.3 Problems: Variational Problems and the Euler-Lagrange Equation

14.1 Walk-Through: The Euler-Lagrange Equation. In this problem you will find the function $y(x)$ that minimizes the integral $\int_1^2 f(y, y', x)dx$ where $f = y^2/x + xy'^2$, subject to the boundary conditions $y(1) = 0$, $y(2) = 1$.

(a) Calculate $\partial f/\partial y$. Remember that this is a partial derivative so you will treat x and y' as constants.

(b) Calculate $\partial f/\partial y'$. (This time treat x and y as constants.)

(c) Calculate $(d/dx)(\partial f/\partial y')$. This is a total derivative, so you will treat x, y and y' as variables, remembering that the derivative of y' is y''.

(d) Plug your formulas into the Euler-Lagrange equation to get a differential equation for $y(x)$.

(e) Find the general solution to that equation. *Hint:* Try guessing a solution of the form $y = x^p$.

(f) Use the boundary conditions to solve for the arbitrary constants and find $y(x)$.

14.2 *[This problem depends on Problem 14.1.]* In Parts (a)–(d), evaluate $\int (y^2/x + xy'^2)dx$ along the given curve from (1, 0) to (2, 1). For instance, if we gave you $y = (x^3 - 1)/7$, you would write $y' = 3x^2/7$ and then:

$$\int_1^2 \left(\frac{\left[(x^3-1)/7\right]^2}{x} + x\left(\frac{3x^2}{7}\right)^2 \right) dx \approx 2.062$$

You should express all your answers as decimals, as we did here.

(a) $y = x - 1$

(b) $y = (x - 1)^2$

(c) $y = 2^{x-1} - 1$

(d) $y = (\ln x)/(\ln 2)$

(e) Calculate the same integral along the path you calculated in Problem 14.1. Show that this path does indeed minimize the integral, at least among these six choices.

(f) Find one more function that connects these two points and evaluate the integral along that function.

14.3 *[This problem depends on the Discovery Exercise (Section 14.2.1)]* In the Discovery Exercise you integrated $y'^2 + 10xy$ from (0, 0) to (1, 1) along several different curves.

(a) Use the Euler-Lagrange equation to show that the curve that minimizes this integral between any two points is $y = (5/6)x^3 + C_1 x + C_2$.

(b) Find the curve that minimizes this integral between (0, 0) and (1, 1).

(c) Compute the integral along that curve and show that your result is lower than any of the integrals you found in the Discovery Exercise.

(d) Sketch the curve. Does it roughly match the prediction you made in the original Exercise?

In Problems 14.4–14.15 find the function $y(x)$ that is a stationary solution for the given integral. If boundary conditions are given plug them in to solve for any arbitrary constants in your solution. If the integrand contains y' but not y consider using the shortcut described in the Explanation (Section 14.2.2.)

14.4 $\int \left(y'^2 - y^2\right) dx$

14.5 $\int \left(y'^2 - y^2 + 2y\right) dx$

14.6 $\int_0^1 \left(y'^2 - y\right) dx$, $y(0) = y(1) = 0$

14.7 $\int x\left(y' + y'^2\right) dx$

14.8 $\int_0^1 \left(y'^2 + xy\right) dx$, $y(0) = 0$, $y(1) = 1$

14.9 $\int_0^1 (1 + x)y'^2 dx$, $y(0) = 0$, $y(1) = 2\ln 2$

14.10 $\int \left(y'^2 - 2xyy'\right) dx$

14.11 $\int_0^2 \left(y'^2 - 2xyy' + y'\right) dx$, $y(0) = 0$, $y(2) = 1$

14.12 $\int_0^{\pi/2} \left(y'^2 + yy' - y^2\right) dx$, $y(0) = 0$, $y(\pi/2) = 2$

14.13 $\int \left(y'^2 + y^2 + y\sin x\right) dx$

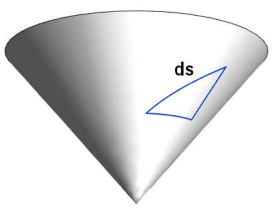
14.14 $\int \left(y' + x^2y'^2\right) dx$

14.15 $\int \sin(y'^2 + 1)dx$ *Hint:* the resulting differential equation is easier than it looks if you think about what it's saying.

14.16 In Problem 14.2 you analyzed the function from Problem 14.1 for a number of different paths, showing that the stationary solution you found does in fact represent a minimum. Choose a problem that you solved from Problems 14.4–14.15 and do the same kind of analysis. Choose a problem that specified beginning and end points, and find at least three functions—in addition to the one you found as the solution—that connect those two points. (You don't need to have done Problem 14.2 to do this.)

14.17 In the Explanation (Section 14.2.2), when we found the shortest path between two points on the cone $z^2 = x^2 + y^2$, we focused on the use of the Euler-Lagrange equation to solve the variational problem. But setting up that problem required finding the length ds of a differential step along the cone. In this problem you will derive that ds geometrically; in Problem 14.18 you will reach the same conclusion algebraically.

Note that you can specify any arbitrary point on this cone by giving its cylindrical coordinates z and ϕ. We therefore begin by representing an arbitrary step ds as a combination of two separate steps: one that changes z without changing ϕ, and one that changes ϕ without changing z.



(a) From the equation $z^2 = x^2 + y^2$ we can see that this cone makes a 45° angle with the horizontal. Explain how we know that. (This fact is going to be important for both the dz and $d\phi$ steps.)

(b) Move in a diagonal line directly up the cone, changing z but not ϕ. If your z-coordinate goes up by dz, how long is the diagonal line you have traversed?

(c) Now move in a circle around the cone, changing ϕ but not z. If your ϕ-coordinate advances by $d\phi$, how long is the arc you have traversed? You may find it easiest to start by writing your answer in terms of ρ, the radius of the cone at whatever height you're at. If you do you should go on to express ρ as a function of z so your final answer only depends on z and $d\phi$.

(d) To find the length of an arbitrary move ds, involving changes to both z and ϕ, use the Pythagorean theorem to put together your two previous results. You can check your answer by making sure it matches the formula we used in the Explanation.

14.18 In the Explanation (Section 14.2.2), when we found the shortest path between two points on the cone $z^2 = x^2 + y^2$, we focused on the use of the Euler-Lagrange equation to solve the variational problem. But setting up that problem required finding the length ds of a differential step along the cone. In Problem 14.17 you derived that ds geometrically; in this problem you will reach the same conclusion algebraically.

(a) Write the distance ds for an arbitrary move in 3D in terms of the Cartesian intervals dx, dy, and dz.

(b) Write formulas for the Cartesian coordinates x, y, and z in terms of the cylindrical coordinates ρ, ϕ, and z.

(c) The cone in that problem was defined by the relationship $z^2 = x^2 + y^2$. Write that equation as a relationship between z and ρ without any x or y in it. Use that relationship to eliminate ρ from your answers to Part (b).

(d) Use your answers to Part (c) to write dx and dy in terms of $d\phi$ and dz. (These expressions will only be valid for intervals along the cone because you used the formula for the cone when you eliminated ρ from the equations.)

(e) Plug these answers into your expression for ds and simplify. You can check your answer by making sure it matches the formula we used in the Explanation.

Problems 14.19–14.22 involve finding the shortest distance between two points on various surfaces. (This is called the "Geodesic problem.") We provide a model for this process in the Explanation, using a cone as the surface.

- Choose an appropriate coordinate system where you can use two variables to specify a point on the surface.
- Find the distance of a step ds along the surface in the coordinate system you are using. You can approach this geometrically as in Problem 14.17, or algebraically as in Problem 14.18.
- Rewrite ds in terms of only one coordinate on a curve.
- You can now express the Geodesic problem as a variational problem. Solve it.

14.19 Prove that the shortest path between two points on the xy plane is a straight line.

14.20 Find the shortest distance between two points on the cone $z^2 = 4(x^2 + y^2)$ (for $z \geq 0$).

14.21 Find a formula for the shortest path between two points on the cylinder defined by $x^2 + y^2 = R^2$. Describe the resulting shape.

14.22 Find a formula for the shortest path between two points on the sphere $x^2 + y^2 + z^2 = R^2$. *Hint:* parametrize your path as $\phi(\theta)$ where ϕ and θ are the angles in spherical coordinates. You should be able to get a first-order differential equation for your path with an arbitrary constant on the right. If you choose your z-axis to pass through the initial point on your path then you can argue that the constant must be zero. That should reduce the equation to something you can solve.

14.23 **A Soap Bubble** Consider a curve connecting two points. The function $y(x)$ that minimizes the surface area you get when you rotate that curve around the y-axis is the shape a soap bubble between two rings would form in the absence of gravity. A small segment of the curve has length $ds = \sqrt{dx^2 + dy^2}$; when rotated about the y-axis this segment sweeps out a cylindrical area $dA = 2\pi x\, ds = 2\pi x\sqrt{dx^2 + dy^2}$.

(a) Find the function $y(x)$ that minimizes this area. Your answer will have two arbitrary constants. *Hint:* you should end up reducing the problem to an integral. You can solve that integral on a computer or look it up in a table.

(b) Solve numerically for the constants if the curve connects $(1, 1)$ and $(2, 0)$. Plot the resulting path.

(c) Is the curve you plotted straight, concave up, or concave down? Explain why that makes sense physically. (If you didn't do the computer part you should still be able to do this part by predicting what the curve should look like.)

14.24 **Exploration: Fermat's Principle** The speed of light depends on what medium it is traveling through, with c (the speed of light in a vacuum) being the fastest possible speed. When light moves from one medium to another it changes both speed and direction.

Consider a light beam that travels from the origin to the point $(2, 2)$. In the following scenarios you will calculate the path this light beam takes. Your guide will be "Fermat's Principle," which tells us in these cases that the light will follow the *fastest path possible* from its starting to its ending position.

(a) First, assume the entire xy plane is a vacuum, so the light travels at c. What path will the beam take from $(0,0)$ to $(2,2)$? You can figure this out with no calculations.

Now assume that from $x = 0$ to $x = 1$ is a vacuum, but from $x = 1$ to $x = 2$ is a medium in which light travels at $0.9c$. The beam will travel in a straight line path to $x = 1$, and then change direction and follow a different straight line to reach the point $(2,2)$.

(b) Will the light reach $x = 1$ at exactly the point $(1,1)$, or slightly below that point, or slightly above it? Explain qualitatively how your answer follows from Fermat's principle.

(c) Write a function for the total time it takes the light beam to complete its journey. The variable in this function will be the y-position of the light beam when $x = 1$. (A drawing will really help here.)

(d) You can find the minimum time by setting dt/dy equal to zero. Write the resulting equation. You can simplify it a bit, but not a whole lot.

(e) Solve the equation. If your answer does not agree with your answer to Part (b), fix one of them.

Now assume that from $x = 0$ to $x = 2$ is a medium that gradually changes, such that the speed of light is $v = c(1 - x/10)$.

(f) Will the light follow a straight line from $(0,0)$ to $(2,2)$, or a concave up curve, or a concave down curve? Explain qualitatively how your solution follows from Fermat's principle.

(g) Write a formula for the distance ds the light travels in an infinitesimal step in its journey. Your formula will depend on dx, the horizontal distance travelled, and on the function $y(x)$ for its path. Use that result to calculate the time dt such a journey makes. Then use that result and calculus of variations to set up a differential equation for the curve $y(x)$. *Hint:* You might end up with a second order ODE, but you can end up with a first order, which is preferable.

(h) Solve the differential equation using the boundary conditions $y(0) = 0$, $y(2) = 2$ to find the path $y(x)$ taken by the light. Sketch the function and make sure its shape matches your answer to Part (f). (You can just ask a computer to plot it and then copy the sketch into your answer.)

14.25 **Exploration: The Brachistochrone** A bead slides down a frictionless track under the force of gravity, falling from rest at $(0,0)$ to arrive at (x_f, y_f) where $y_f < 0$. The "brachistochrone" is defined as the curve $y(x)$ that will get the bead to its final destination in the shortest possible time.

(a) A straight line would be the *shortest* path, but it wouldn't be the fastest one. Explain why not and sketch qualitatively what the brachistocrone curve should look like.

(b) Using conservation of energy, find the speed of the bead at a point (x, y) along the track.

(c) In a short interval of time the bead travels a distance $ds = \sqrt{dx^2 + dy^2}$, requiring a time of $dt = ds/v$. You need to express the time dt in one of two ways: as a function of x, y, $y'(x)$, and dx, or as a function of x, y, $x'(y)$, and dy. Explain why we prefer the latter.

(d) Write an integral that represents the total time for the bead to fall, and use the techniques of this section to set up a first order differential equation for the optimal curve $x(y)$.

(e) Show that the following parametric curve (where k is any constant) is a solution to your differential equation: $x = k(\theta - \sin\theta)$, $y = k(\cos\theta - 1)$. *Hint:* for a parametrically expressed curve you can calculate dx/dy as $(dx/d\theta)/(dy/d\theta)$.

A curve with this parametric representation is called a "cycloid," so you just proved that the brachistochrone curve is a cycloid.

14.26 *[This problem depends on Problem 14.25.]* Consider a particle sliding from the origin to the point $(1, -1)$. Assume all quantities are in SI units and take $g = 10$.

(a) Calculate the constant C for the brachistochrone curve connecting those points. Plot the resulting curve.

(b) Calculate the time required for a particle to slide along that curve.

(c) Compare that time to the time for a particle to slide along a straight line connecting the same two points.

14.3 Why the Euler-Lagrange Equation Works

As with many formulas, you can use the Euler-Lagrange equation without ever knowing where it comes from. But going through the derivation once or twice, even if you can't reproduce it years later, gives you a deeper appreciation for the nature of the problem you're solving.

14.3.1 Explanation: Why the Euler-Lagrange Equation Works

You know that every local minimum or maximum of a one-variable function $f(x)$ must occur at a point where $f'(x)$ is zero.[3] Can you explain why? A drawing can be very convincing, but we need something more mathematical.

Here is one way you can frame the argument. Suppose we claim that $f(x)$ reaches a minimum at a particular value $x = x_0$. That means by definition that changing x by a small amount—in either direction—will *not* cause f to decrease.

We use dx to represent that small change in x, and $dy = f'(x)dx$ to represent the resulting change in the function f. With that notation, the argument goes like this. If $f'(x_0)$ is negative, then increasing x (moving to the right) will cause f to decrease. Conversely, if $f'(x_0)$ is positive, then *decreasing* x will cause f to decrease. In neither case have you found a minimum value for the function! We conclude that f can only attain a minimum at x_0 if $f'(x_0) = 0$.

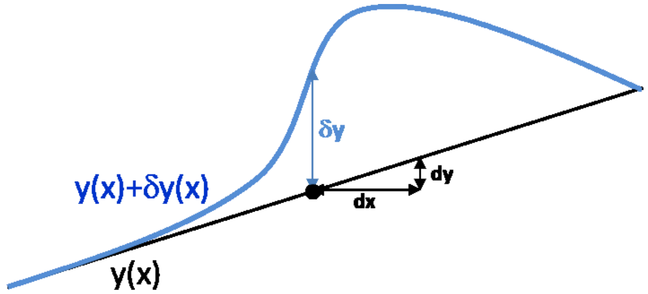
Before you read further, try the following exercise: rewrite the previous two paragraphs for a variational problem. The goal here is not to arrive at the final answer (the Euler-Lagrange equation), but to frame the question properly. One cautionary note: in the above discussion, y and f both represented the same function. But throughout the rest of this section, y will represent a curve and f will represent a function that is defined along that curve.

…pause while you write…

Hopefully you started with something like this: "Suppose we claim that $I = \int f(y, y', x)dx$ reaches a minimum along a particular curve $y(x)$. That means by definition that changing $y(x)$ by a small amount—in any possible way—will not cause I to decrease."

If you went on from there to discuss dy and df/dy, you're doing great. But later you run into some confusion because two different kinds of dy show up in the same argument. We're going to introduce some new notation to distinguish between them.

1. Good old dy means "I am moving along a curve by a small dx (moving from left to right if dx is positive) and seeing the resulting change in y." So dy/dx is the slope of the curve as always.
2. The new $\delta y(x)$ means "At each x-value I am changing the value of y by changing the curve itself." That causes a change δI, and finding a formula for that change is going to be one of our main tasks in this section.



[3] or is undefined, but we will confine our discussion to differentiable functions

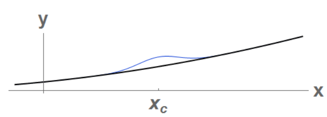
Don't confuse this new δ with the old ∂ for a partial derivative, which we will also of course be using!

With that notation in place, the rest of the argument looks much the same as it did the first time. If a particular infinitesimal change $\delta y(x)$ results in a negative δI then that change will decrease I, so we can't be at a minimum. And if it results in a positive δI then the opposite change $-\delta y(x)$ will decrease I, so we still can't be at a minimum. The optimum curve $y(x)$ must therefore have the property that any tiny change $\delta y(x)$ away from it will give $\delta I = 0$.

You can make these definitions more rigorous by defining $\delta y(x) = \epsilon\eta(x)$ where ϵ is a number and $\eta(x)$ is an arbitrary function. Then we require $dI/d\epsilon = 0$ for all smooth functions $\eta(x)$ subject to the condition that $\eta = 0$ at both boundaries. That last condition occurs because $y(x)$ must meet the given boundary conditions so it can't vary at the boundaries. It's straightforward to rewrite all our derivations in this section in terms of ϵ and η but we find the δy notation easier to follow.

What we are going to do for the rest of this section is try to convince you that $\delta I = 0$ for any small variation $\delta y(x)$ if and only if $\partial f/\partial y - (d/dx)(\partial f/\partial y') = 0$. In fact we're going to make three different arguments: first a visual hand-wave, then a derivation involving integration by parts, and finally (in Problem 14.31) a derivation based on Riemann sums. But make sure you first understand what we've presented so far: what δI represents and why it has to be zero for any small $\delta y(x)$ if $y(x)$ is a stationary solution. Without that, the rest of this section won't mean much.

A Hand-Waving Argument



$y(x)$ (black) and $y + \delta y$ (blue)

We begin with a curve $y(x)$ and a function $f(y, y', x)$ defined along that curve. Then we change $y(x)$ by a small δy at a particular $x = x_c$. Of course, changing $y(x)$ at *only* that point would make the curve discontinuous, so we don't do that; we bend the curve up, as shown above. Then we ask the question, how does this change affect the value of our integral $I = \int f(y', y, x)dx$?

Most directly it changes y. If $\partial f/\partial y$ is positive, that signifies that increasing y at a particular x-value will increase f at that value, thus increasing the total integral I. If $\partial f/\partial y$ is negative then this effect would decrease I.

But our bump in the curve also changes y'. Specifically, y' increases to the left of x_c and decreases to the right of x_c. If $\partial f/\partial y'$ is the same on both sides of x_c then the net effect of changing y' will cancel out. On the other hand, if $\partial f/\partial y'$ is larger on the right than on the left, then the effect of a positive δy will be to decrease $\int f\, dx$. (Make sure you see that.) So the change in the integral will be negative if $(d/dx)(\partial f/\partial y')$ is positive, and vice versa.

Putting all this together it at least seems plausible that the total change δI from a small change $\delta y(x)$ will look something like $(\partial f/\partial y) - (d/dx)(\partial f/\partial y')$. Of course this is not a proof. There could be other numerical factors, for example. But this hopefully gives you some intuition for why those particular two terms appear in the Euler-Lagrange equation and why they have the signs they do. This argument can be made more rigorous by considering the integral as a Riemann sum and then taking the limit as $dx \to 0$. See Problem 14.31.

As a final note, remember that the above argument centered on a change at a particular $x = x_c$. A minimum means that no change to $y(x)$ *anywhere* will cause I to decrease. So Euler-Lagrange is a differential equation, requiring that $(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0$ everywhere along the curve.

A Real Derivation Using Integration by Parts

We begin once again with the following question. We have evaluated $I = \int f(y, y', x)dx$ along a given curve $y(x)$. Now we introduce a change $\delta y(x)$ to the curve. What effect does that have on the integral?

Consider the f-value at one particular x-value. Here y has gone up by some particular δy, and y' has gone up by $\delta y'$. We can compute the resulting change in f from the chain rule.

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

That calculation represents the change in f at one particular x-value. To find the total change in I, we add up the changes along the curve.

$$\delta I = \int_{x_0}^{x_f} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right) dx$$

It's painfully easy to get lost in the symbols, so keep asking yourself what all the pieces mean. For instance, consider the second term in that integrand. $\partial f/\partial y'$ asks the question—at a particular x-value—"if I increased the slope of my curve, all other things being equal, how fast would that increase the function f?" We multiply that by $\delta y'$: "when we changed our old curve to our new curve, how much did the slope change at this point?" The product gives us part (but not all) of the change in f caused by changing the curve at this particular x-value. When we add the contribution represented by the first term and integrate across the entire domain, we get the total change in I.

We next replace $\delta y'$ with $(d/dx)(\delta y)$. That substitution is not entirely obvious; think about it for a moment, remembering that $\delta y'$ means the *change in the slope* caused by δy. See Problem 14.27.

$$\delta I = \int_{x_0}^{x_f} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}\delta y \right) dx \qquad (14.3.1)$$

That substitution allows us to use integration by parts on the second term in the integrand.

$$u = \frac{\partial f}{\partial y'} \qquad dv = \left(\frac{d}{dx}\delta y\right) dx$$

$$du = \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx \qquad v = \delta y$$

$$\int \frac{\partial f}{\partial y'}\left(\frac{d}{dx}\delta y\right) dx = \frac{\partial f}{\partial y'}\delta y - \int \delta y \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx$$

Plug that back into Equation 14.3.1 and rearrange.

$$\delta I = \left.\frac{\partial f}{\partial y'}\delta y\right|_{x_0}^{x_f} + \int_{x_0}^{x_f} \delta y \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) dx$$

Remember that a variational problem fixes the endpoints at $y(x_0) = y_0$ and $y(x_f) = y_f$. That means that any variation δy that you consider has to go to zero at both ends, so the term outside the integral is zero.

Now it remains to say what must be true for δI to equal zero. In general an integral can equal zero without the integrand being zero everywhere. But in this case we have to be sure that there is no possible variation $\delta y(x)$ for which $\delta I \neq 0$. The only way $\int \delta y$<*stuff*>dx can be zero for all possible functions δy is if <*stuff*> equals zero everywhere. (This is the mathematical equivalent of the argument we made at the end of our hand-wave: the integral I can only reach a minimum if no change in $y(x)$, anywhere along the curve, can result in a decrease in f.)

We conclude once again that the stationary solutions occur where $(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0$.

14.3.2 Problems: Why the Euler-Lagrange Equation Works

14.27 Our derivation of the Euler-Langrange equation included a step in which we replaced $\delta y'$ with $(d/dx)(\delta y)$. In this problem you will consider that replacement.

(a) Draw a function $y(x)$. (Make it reasonably simple, but not as simple as a line.) Now draw a second function $y + \delta y$ where δy is the same for all x-values. What is $(d/dx)(\delta y)$—the rate of change of δy as you move from left to right—in this case? What is $\delta y'$—the change of the slope from the first function to the second—in this case?

(b) Redraw your same $y(x)$ function. Then draw a function $y + \delta y$ where δy starts at zero on the left, and continually increases as you go to the right. Is $(d/dx)(\delta y)$ positive or negative, and how can you tell from your drawing? Is $\delta y'$ positive or negative, and how can you tell from your drawing?

(c) Explain in your own words why, in general, $\delta y' = (d/dx)(\delta y)$.

14.28 The Explanation (Section 14.3.1) presented a derivation using integration by parts for the stationary solutions to the problem of extremizing $\int f(y, y', x)dx$ subject to fixed boundary conditions at the two ends. The result was the Euler-Lagrange equation.

(a) Following a similar argument, derive an equation for the stationary solutions to the problem of extremizing $\int f(y, y', y'', x)dx$ subject to fixed boundary conditions on y and y' at the two ends.

(b) Which step in your derivation would not have been valid if we had only specified y at the boundaries instead of y and y'?

14.29 The Explanation (Section 14.3.1) presented a derivation using integration by parts for the stationary solutions to the problem of extremizing $\int f(y, y', x)dx$ subject to fixed boundary conditions at the two ends. The result was the Euler-Lagrange equation. Generalize this argument to derive the Euler-Lagrange equations for the case of two curves, $\int f(y, y', z, z', x)$, where y and z are both functions of x with specified values at the boundaries. The formulas you derive in this problem will be used in many problems in Section 14.4.

14.30 *[This problem depends on Problem 14.29.]*

(a) Write a pair of coupled differential equations for the stationary solutions that minimize the integral $\int (z'^2 + 2y'^2 + 3yz)dt$.

(b) If you've studied linear algebra you might know how to solve those equations, but here we'll point you towards one solution: $y(t) = \sin(kt)$, $z(t) = -\sqrt{2}\sin(kt)$. Find the value of k for which these represent a stationary solution to the original integral.

14.31 In this problem you'll derive the Euler-Lagrange equation by rewriting $I = \int f(y, y', x)dx$ as a Riemann sum $I = \lim_{\Delta x \to 0} R\,\Delta x$, where

$$R = \sum_{i=0}^{N-1} f[y(x_i), y'(x_i), x_i]$$

with

$$x_i = x_0 + i\Delta x, \; N = \frac{x_f - x_0}{\Delta x}$$

First you'll find the critical points for R, and in the limit $\Delta x \to 0$ these will becomes the stationary solutions for I. Throughout the problem we'll use y_i and y_i' to mean y and y' evaluated at x_i.

(a) For a finite N this is a multivariate calculus optimization, not a calculus of variations problem. What are the variables that you are varying in order to find a critical point for R? *Hint*: there are N of them.

(b) Consider the effect of increasing y_i by an amount δy while leaving all of the other y values constant. Ignoring the effect this has on y' for the moment, how much does this change R? Your answer will depend on $\partial f/\partial y_i$, meaning the partial derivative of f with respect to y, evaluated at the point x_i.

Now consider the effect that increasing y_i has on y'. In the Riemann sum we can approximate y_i' with $[y_{i+1} - y_{i-1}]/(2\Delta x)$.

(c) Which two y' values are affected when you increase y_i by δy?

(d) Figure out how much each of those two values are affected and put them together to find the total change in R that occurs because of y' when you increase y_i by δy.

(e) In the limit $\Delta x \to 0$ your answer to Part (d) can be written as δy times a derivative with respect to x. Rewrite it that way. *Hint*: look at how we wrote y_i' as a guide.

(f) Combine your answers to Parts (b)–(d) to find the total change in R resulting from increasing y_i by δy.

(g) A critical point for a multivariate function occurs at a point where infinitesimal changes in any of the variables leads to zero change in the function. You just considered the effect of a change in y_i on R. If you assume that δR resulting from a change in any of the y_i equals zero, your answer becomes a differential equation for y. Write that equation.

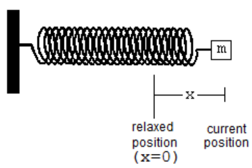
Note that this derivation isn't quite complete because the formula we used for y' isn't valid at the two endpoints. Since δy must be zero at the endpoints to satisfy the boundary conditions, however, the derivation is valid.

14.4 Special Application: Lagrangian Mechanics

A century after Newton presented his three laws of motion, Joseph-Louis Lagrange introduced an alternative. Lagrange's approach and Newton's give the same prediction in any situation, and you can use either one to derive the other. But "Lagrangian mechanics" starts from a different set of postulates and, in many cases, is more convenient for calculations than $F = ma$.

Lagrangian mechanics offers particular benefits when all the forces involved are "conservative" (see Chapter 8). In such a case you can describe a potential energy for every possible configuration of particles, which is equivalent to specifying all the relevant force laws. In this section we will consider only conservative forces.

The Principle of Least Action



To solve for the motion of a mass on an ideal spring, you might write the differential equation that represents Newton's second law: $F = m\ddot{x} = -kx$. (We're going to use the dot notation for time derivatives a lot in this section. Remember that $\ddot{x}$ means d^2x/dt^2.) Alternatively, because the forces involved are conservative, you might write the differential equation that represents conservation of energy: $KE + U = (1/2)m\dot{x}^2 + (1/2)kx^2 = C$. These two approaches are based on fundamentally different premises, but they lead to the same final solution: $x(t) = A \sin\left(\sqrt{k/m}\, t\right) + B \cos\left(\sqrt{k/m}\, t\right)$. You choose one approach or the other based on mathematical convenience.

Lagrangian mechanics starts from a premise that is quite different from either $F = ma$ or conservation of energy.

The Equation of Motion in Lagrangian Mechanics

1. The "Lagrangian" of an object is its kinetic energy minus its potential energy: $L = KE - U$.
2. "Action" is the time integral of an object's Lagrangian as it moves along a given trajectory: $S = \int L\, dt$.
3. The "Principle of Least Action" says that in moving from position $x(t_0) = x_0$ to position $x(t_f) = x_f$ the object will follow the trajectory $x(t)$ that minimizes the action.[a]

Find the trajectory that minimizes an integral? Hey, it's a variational problem! We therefore approach it with the Euler-Lagrange formula, this time applied to a function $L(x, \dot{x}, t)$ instead of our old $f(y, y', x)$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

[a] Strictly speaking the principle says that the trajectory will be a stationary solution, which could be a minimum or a maximum. In practice it's almost always a minimum. The principle is sometimes called the "Principle of Stationary Action." It's also sometimes called "Hamilton's Principle."

We can illustrate the use of this equation using the same example we solved with Newtonian methods above.

EXAMPLE **An Object on a Spring**

Problem:
Find the position $x(t)$ of a mass m attached to an ideal spring with spring constant k.

Solution:
The potential energy of a mass on an ideal spring is $U = (1/2)kx^2$. The Lagrangian is $KE - U$.

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \rightarrow \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -kx$$

So the Euler-Lagrange formula in this case becomes $m\ddot{x} + kx = 0$. This is the same differential equation we got from $F = ma$ so of course it leads to the same solution, $x = A\sin\left(\sqrt{k/m}\,t\right) + B\cos\left(\sqrt{k/m}\,t\right)$.

Generalized Coordinates

The example above led to the same equation we got from Newton's second law, only with more work on our part. You'll show in Problem 14.32 that it's simple to derive Newton's second law from the principle of least action. So, why are we doing this?

There are a number of reasons why Lagrangian mechanics is useful. One is that it generalizes to systems where Newton's laws don't apply. You can write down Lagrangians for relativistic particles, for fields, or even for the curvature of spacetime in general relativity, and the equations of motion in all those cases follow from the principle of least action. The postulates of quantum field theory are most easily stated in terms of Lagrangians (although they are not the same as the principle of least action, which doesn't hold for quantum systems).

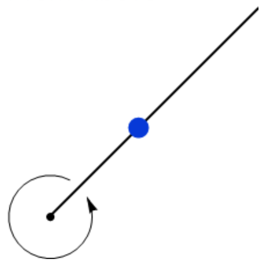
For classical systems of objects, what makes Lagrangian mechanics useful is that you can write the Lagrangian in terms of "generalized coordinates"—any numbers that describe the state of the system—not just Cartesian coordinates. As long as you can express the kinetic and potential energy of your system in terms of your generalized coordinates and their time derivatives (sometimes called "generalized velocities"), you can write the Lagrangian and solve the Euler-Lagrange equations. In many cases this makes complicated problems much easier than they would be with Newton's laws[4].

[4] You can write Newton's laws for any set of generalized coordinates, but the equation of motion will not in general be $F = ma$. The beauty of Lagrangian mechanics is that the Euler-Lagrange equation in the form we've written it applies no matter what coordinates you use.

EXAMPLE

A Bead on a Rotating Rod

Problem:
A bead of mass m is placed on a frictionless rod that is rotating horizontally at a constant angular speed ω. Find the distance $\rho(t)$ of the bead from the center of the rotation.



Solution:
We will express this situation in polar coordinates,[b] because the problem determines $\phi(t)$ explicitly and asks for $\rho(t)$. Because the rotation is horizontal there is no potential energy. The kinetic energy is $(1/2)mv^2$. (Since the motion of the rod is fixed by some external force we just consider our system to be the bead.) The radial component of velocity is $\dot{\rho}$. The tangential component is perhaps less obvious, but if you remember that $\omega = \dot{\phi}$ is the angular velocity you can convert that to tangential velocity with the formula $v_{tangential} = \omega\rho$. That leads to the Lagrangian $L = (1/2)m\dot{\rho}^2 + (1/2)m\omega^2\rho^2$. (You could also derive that second term from the formula for kinetic energy of rotation, $KE = (1/2)I\omega^2$, where the moment of inertia is $I = m\rho^2$.)

$$\begin{aligned} \frac{dL}{d\dot{\rho}} &= m\dot{\rho} \\ \frac{d}{dt}\left(\frac{dL}{d\dot{\rho}}\right) &= m\ddot{\rho} \\ \frac{dL}{d\rho} &= m\omega^2\rho \\ m\ddot{\rho} - m\omega^2\rho &= 0 \end{aligned}$$

The solution is $\rho(t) = Ae^{\omega t} + Be^{-\omega t}$. Unless the initial conditions are perfectly fine-tuned to set $A = 0$ the first term will come to dominate and the bead will move away from the origin exponentially with time.[c]

As a final note, we know that ω has units of one over time, so the arguments of the exponentials are unitless, as they should be. (When one of the authors first solved this problem he made a mistake and got $e^{-\omega^2 t}$, but immediately spotted the error when he checked units.)

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[b] You may be used to using r and θ for polar coordinates. We use the letters ρ and ϕ but they mean the same thing: distance from the origin and angle going counterclockwise from the positive x axis.

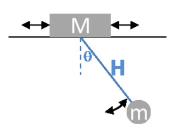
[c] If you're a first or second year physics student, this is caused by interactions between the bead and the rod because there is no such thing as centrifugal force. If you are a junior physics major or beyond, it's just due to the centrifugal force. `http://xkcd.com/123`

The example above applies the Euler-Lagrange equation in a simple setting; the example below demonstrates more advanced techniques that you will need in many of the problems. Note three differences in particular.

1. The choice of coordinates was obvious in the previous example. The following example better illustrates how the choice of coordinates can arise from the constraints inherent in the situation.
2. We wrote down the generalized velocities in the previous example with some arguments from introductory physics. Below we derive our velocities from the equations that convert from generalized to Cartesian coordinates. (The second approach always works, but the first is faster and easier when you can do it.)
3. The example below involves minimizing an integral that depends on *two* functions instead of just one. In such cases you write down the Euler-Lagrange equation separately for each function. *How* you do that is demonstrated in our solution; *why* you can do that is derived in Problem 14.29.

EXAMPLE A Sliding Pendulum

Problem:
A pendulum of length H and mass m is hung from a block of mass M that is free to slide horizontally, as shown below. Choose an appropriate set of generalized coordinates for this system and find the equations of motion. (We are using H for the pendulum length so it doesn't get confused with the Lagrangian L.)



Solution:
The simplest choice of generalized coordinates is the horizontal position of the block, X, and the angle θ of the pendulum, which we define to be zero when it is straight down. It's easiest to express the energy in terms of the Cartesian coordinates of the pendulum bob, x and y. The kinetic energy is $(1/2)M\dot{X}^2 + (1/2)m\dot{x}^2 + (1/2)m\dot{y}^2$ and the potential energy is mgy. (The block has no potential energy.) Next we have to relate x and y to X and θ so we can get this all in terms of our generalized coordinates.

You might wonder why we don't just use x, y, and X. That would be too many degrees of freedom; x and y are constrained by the fact that the bob always stays at the end of the pendulum string. The simplest way to deal with that is to use two generalized coordinates that describe the motion with no additional constraints required.

If we set $y = 0$ at the top of the pendulum then $y = -H\cos\theta$. For the x coordinate we need to account for the position of the block: $x = X + H\sin\theta$. From these we get $\dot{y} = H(\sin\theta)\dot{\theta}$ and $\dot{x} = \dot{X} + H(\cos\theta)\dot{\theta}$ and from those we get the Lagrangian. (Notice that the potential energy is $-mgH\cos\theta$ and the Lagrangian subtracts the potential energy, so it shows up with a plus sign.)

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{X}^2 + mH(\cos\theta)\dot{X}\dot{\theta} + \frac{1}{2}mH^2(\cos^2\theta)\dot{\theta}^2 + \frac{1}{2}mH^2(\sin^2\theta)\dot{\theta}^2 + mgH\cos\theta$$
$$= \frac{1}{2}(M+m)\dot{X}^2 + mH(\cos\theta)\dot{X}\dot{\theta} + \frac{1}{2}mH^2\dot{\theta}^2 + mgH\cos\theta$$

For the most part this looks like the Lagrangian for a freely sliding system with mass $M+m$ plus the Lagrangian for a rotating pendulum of mass m. All of the interaction between the two motions comes about because of the second term, which couples the two velocities. To see the effect that has we write the Euler-Lagrange equations for both our independent variables.

$$\begin{aligned}\frac{\partial L}{\partial \dot{X}} &= (M+m)\dot{X} + mH(\cos\theta)\dot{\theta} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) &= (M+m)\ddot{X} + mH(\cos\theta)\ddot{\theta} \\ &\quad - mH(\sin\theta)\dot{\theta}^2 \\ \frac{\partial L}{\partial X} &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}} &= mH(\cos\theta)\dot{X} + mH^2\dot{\theta} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= mH(\cos\theta)\ddot{X} - mH(\sin\theta)\dot{X}\dot{\theta} \\ &\quad + mH^2\ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mH(\sin\theta)\dot{X}\dot{\theta} - mgH\sin\theta\end{aligned}$$

Plugging this into the Euler-Lagrange equations and cancelling some constants we get the equations of motion for the system.

$$\begin{aligned}\left(\frac{M}{m}+1\right)\ddot{X} + H(\cos\theta)\ddot{\theta} - H(\sin\theta)\dot{\theta}^2 &= 0 \\ (\cos\theta)\ddot{X} - (\sin\theta)\dot{X}\dot{\theta} + H\ddot{\theta} + (\sin\theta)\dot{X}\dot{\theta} + g\sin\theta &= 0\end{aligned}$$

As a reality check notice that all the terms in both equations have units of distance over time squared. (It doesn't matter that the two equations have the same units as each other, but if two terms in one of the equations had different units from each other we would know we'd made a mistake.)

Of course you're not likely to be able to solve the equations of motion we just derived for the sliding pendulum, but you can always ask a computer to do that, numerically if not analytically. What Lagrangian mechanics allowed you to do was go from the physical description to a set of equations that you can give to a computer. See Problem 14.40.

As a final note we should say that you can still sometimes use the shortcut we explained in Section 14.2.2 for variational problems that involve $\dot{x}$ but not x. If the Lagrangian for one of your generalized coordinates q involves $\dot{q}$ but not q itself then the Euler-Lagrange equation for that coordinate can be written $\partial L/\partial \dot{q} = C$ where C is an arbitrary constant. (We could have used this shortcut for X in the sliding pendulum problem above but since $\ddot{X}$ shows up in the other Euler-Lagrange equation this wouldn't have made the equations any easier to work with.)

14.4.1 Problems: Lagrangian Mechanics

14.32 Consider a particle with kinetic energy $(1/2)m\dot{x}^2$ and potential energy $U(x)$. Recalling that $F = -dU/dx$, prove that the Euler-Lagrange equation for the motion of the particle is equivalent to Newton's second law.

14.33 Using the equation $(1/2)m\dot{x}^2 + (1/2)kx^2 = C$ and the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$, derive the solution $x(t)$ for a mass on a spring pulled out to a distance x_0 and released. Check that your answer matches the one we got.

14.34 A ball with mass m travels under the influence of a constant gravitational force $F = mg$. Use Lagrangian mechanics to write the equation of motion for this mass, and then solve that equation to show that the resulting motion is a quadratic function $y(t)$.

14.35 A comet with mass m is traveling under the influence of Earth gravity, a force $F = -GM_E m/r^2$. Use Lagrangian mechanics to write the equation of motion for this comet. You do not need to solve the resulting differential equation.

14.36 A massless spring with spring constant k hangs down from the ceiling. At the end of the spring is a mass m.

(a) Use Lagrangian mechanics to write the equation of motion for this mass. You do not need to solve the resulting differential equation (although you can, if you have gone through Chapter 1).

(b) Use Newtonian mechanics to write the equation of motion for the same mass. Hopefully you'll get the same answer!

14.37 Use Lagrangian mechanics to write the equation of motion for a pendulum consisting of a ball of mass m hung from a massless string of length H. You do not need to solve the resulting differential equation.

14.38 In the example on Page 17 we derived and solved the equations of motion for a bead on a horizontal, rotating rod. Do the same for a bead on a rod that is rotating vertically with constant angular velocity ω.

14.39 In the example on Page 17 we derived and solved the equations of motion for a bead on a horizontal, rotating rod.

(a) Redo the problem assuming the rod is slowing down: $\omega = \sqrt{2}/t$. Find the general solution $\rho(t)$.

(b) Now assume that at time $t = 1$ the system begins, with the bead at $\rho = 1$ with radial speed $\dot{\rho} = 0$. Solve for the arbitrary coefficients and find $\rho(t)$.

14.40 In the example on Page 18 we found the equations of motion for a sliding pendulum. Solve those equations to find the motion if the system starts at rest with the pendulum pulled up to an angle of $\pi/4$. Take $H = 1$ m and $M/m = 2$. Have the computer draw the block and pendulum at a series of times to show what the resulting motion looks like.

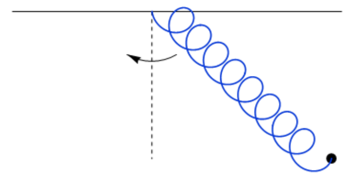
14.41 In the example on Page 18 we found the equations of motion for a sliding pendulum. Now suppose the block is not free to move but is pushed back and forth: $X = A\sin(\omega t)$.

(a) Find the Euler-Lagrange equation for $\theta(t)$.

(b) For small oscillations $\theta \approx 0$ you can approximate this differential equation with a linear one. Make that approximation and solve the resulting differential equation.

(c) The equations of motion you wrote in Part (a) should describe this system accurately, but your solution in Part (b) is only valid if θ remains small. What physical circumstances would make this a reasonable, or an unreasonable, approximation?

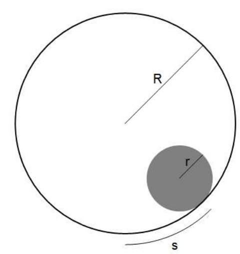
14.42 The picture below shows a ball hanging from a massless spring that is free to swing back and forth like a pendulum.



Assume the ball has mass m and the spring has spring constant k and equilibrium length H. Choose an appropriate set of generalized coordinates and find the equations of motion for the ball. You do not need to solve the resulting differential equations.

14.43 *[This problem depends on Problem 14.42.]* Solve the equations of motion to find the motion if the system starts at rest with the pendulum pulled up to an angle of $\pi/4$ and the spring at its equilibrium length. Take $H = 1$ m, $k = 5$ N/m, and $m = 1$ kg. Have the computer draw the pendulum at a series of times to show what the resulting motion looks like.

14.44 A uniform, solid cylinder of mass m and radius r is rolling inside a hollow cylinder with larger radius R. Let s be the arclength from the small cylinder's current position to the bottom. The larger cylinder does not move.



(a) Find the Euler-Lagrange equation of motion for s. *Hint:* you will need to look up (or calculate) the moment of inertia of a uniform solid cylinder about its axis. You do not need to solve the resulting differential equation.

(b) What is the frequency of small oscillations of this system?

14.45 A block of mass m is sliding on the inside of a frictionless, hollow, hemispherical bowl of radius R. Choose an appropriate set of generalized coordinates and write the equations of motion for the block. You do not need to solve the resulting differential equations. (The bowl does not move.)

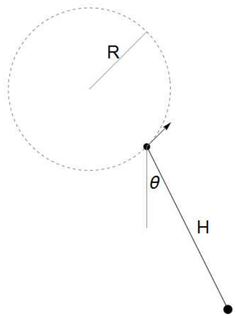
14.46 An iron block of mass m is sliding on the inside of a frictionless, hollow, hemispherical bowl of radius R. A uniform magnetic field exerts a constant force $F_B\hat{i}$ on the block. (The bowl does not move.)

(a) In general magnetic forces cannot be associated with a scalar potential energy, but in this case the magnetic force can. Find the potential energy for that force.

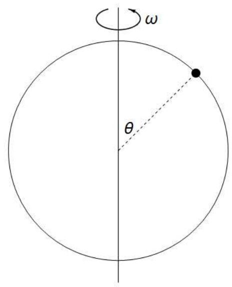
(b) Choose an appropriate set of generalized coordinates and write the equations of motion for the block. You do not need to solve the resulting differential equations.

14.47 A block of mass m is sliding on the inside of a frictionless, hollow cone with the vertex at the bottom, height H, and upper radius R. Choose an appropriate set of generalized coordinates and write the equations of motion for the block.

14.48 A pendulum of length H is suspended from a point that is being moved about a vertical circle of radius R with angular speed ω. Find the equation of motion for the pendulum's angle $\theta(t)$.



14.49 A bead of mass m is strung on a wire that is bent into a vertical circle of radius R. The circle is spun around its vertical diameter at constant angular speed ω.



(a) Find the equation of motion for the angle of the bead on the wire. You do not need to solve the resulting differential equation.

(b) Your differential equation should have two or three equilibrium values of θ, depending on the value of ω. Find them, and explain physically why each one is an equilibrium point for the bead.

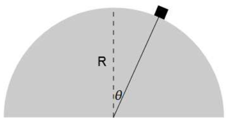
(c) Identify each of these equilibrium angles as stable, unstable, or stable under certain (specified) circumstances.

(d) A stable equilibrium can lead to oscillatory behavior. Find the frequency of oscillation around one stable equilibrium you identified, assuming these oscillations are small.

14.50 **Exploration: Lagrange Multipliers** If you want to minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$ you have two main choices. You can use the constraint to eliminate one variable, write f as a single-variable function, and optimize it. Alternatively you can use a Lagrange multiplier to solve the problem in terms of both variables. (See Chapter 4.) The same is true for variational problems. If you want to minimize $\int f(y, y', z, z', x)dx$ subject to the constraint $g(y, z) = 0$ you can either use the constraint to eliminate y or z from the problem or you can use the following modified form of the Euler-Lagrange equations.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0$$
$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) - \frac{\partial f}{\partial z} - \lambda\frac{\partial g}{\partial z} = 0$$
$$g = 0$$

For most calculus of variations problems it's just as easy to eliminate variables and not worry about λ. In Lagrangian mechanics, however, the new variable λ gives you the force that holds the object on the constrained surface. As an example, consider a block sliding down a hemispherical mound of radius R. The block starts at rest at the top and is given an infinitesimal nudge to get it moving.



(a) First use a single generalized coordinate θ for the block's angle as it slides down. Write the Euler-Lagrange equation for θ.

(b) This equation has no simple solution. At a glance it looks like you could reasonably approximate the motion by replacing $\sin\theta$ with θ. Explain why this wouldn't make sense in this case.

(c) Now reconsider the problem with the generalized coordinates r (radial distance) and θ and the constraint $r - R = 0$. Find the Euler-Lagrange equations for r and θ. In writing the kinetic and potential energy treat r as a free variable; the constraint will come in through λ.

(d) With the constraint $r = R$ you can set $\dot{r} = \ddot{r} = 0$. Use that to get an expression for λ that only depends on θ, $\dot{\theta}$, and constants.

(e) Use conservation of energy to express $\dot{\theta}$ as a function of θ. Plug this into your earlier equation to find λ as a function of θ.

(f) The constraint force λ is in this case the normal force. Find λ at $\theta = 0$ and explain why your answer makes sense.

(g) Use your answer for $\lambda(\theta)$ to predict when the block will lose contact with the surface of the hemisphere.

14.5 Additional Problems

In Problems 14.51–14.53 find the function $y(x)$ that is a stationary solution for the given integral. If boundary conditions are given plug them in to solve for any arbitrary constants in your solution. If the integrand contains y' but not y consider using the shortcut described in Section 14.2.

14.51 $\int y'^2 dx$, $y(0) = 0$, $y(1) = 1$

14.52 $\int \sqrt{y'^2 + x^2}\, dx$

14.53 $\int \left(xy'^2 + y^2/x\right) dx$, $y(1) = 0$, $y(2) = 4$

14.54 Prove that a stationary solution to $\int f(y')dx$ between any two endpoints is always a straight line for any smooth function $f(y')$. *Hint:* If you're stuck pick a couple of simple examples of $f(y')$ and solve those first.

14.55 **(a)** Prove that for the integral $\int_{x_0}^{x_f} y' f(y)dx$ any function $y(x)$ is a stationary solution regardless of the function $f(y)$. *Hint:* If you're stuck pick a couple of simple examples of $f(y)$ and solve those first.

(b) To see why this happened, start with a specific example: $\int_0^1 y'y^2 dx$ subject to the boundary conditions $y(0) = 0$, $y(1) = 1$. Evaluate this integral. You should be able to get a specific numerical answer *without* knowing what the function $y(x)$ is. *Same hint as above*: if you're stuck try doing this for $y(x) = \sin(\pi x/2)$. You should be able to see that the basic process you use for that case can work for any $y(x)$ that meets the boundary conditions.

(c) Now generalize that result to explain why all functions $y(x)$ are stationary solutions of the integral $\int_{x_0}^{x_f} y' f(y)dx$ for any function $f(y)$.

14.56 Prove that the shortest path between two points on the plane $ax + by + cz = 0$ is a straight line.

14.57 Find a formula for the shortest path between two points on the parabolic cylinder $y = kx^2$. You should get as far as writing $z(x)$ as an integral by hand, but you're welcome to turn that slightly messy integral over to a computer.

14.58 Cecelia is in her lifeguard stand at the edge of the water. Take her stand to be at the origin and take the y axis to point directly out to sea. The water is shallow enough that she runs through it to rescue swimmers, but the deeper she goes the slower she runs so her speed is $v = v_0 - ky^2$.

(a) Write a differential equation for the quickest path she can take to reach a drowning swimmer at position (W, H). (These are 2D coordinates because we're assuming the swimmer is at the surface. Take W and H to be positive.)

(b) Without solving the differential equation (yet), sketch what the path should look like. Explain how you know if it will be straight, curved up, or curved down.

(c) Take the swimmer's position to be $(1, 1)$, $v_0 = 1$, and $k = 0.5$. Solve the equation you found and plot the optimal path. If it doesn't match your expectation figure out whether your logic or your calculations went wrong.

14.59 The electrostatic potential from a point charge q at a point P is kq/r, where k is a constant and r is the distance from P to the point charge. For a continuous charge distribution you find the potential at P by breaking the charge into pointlike pieces and integrating. A string of uniform charge per unit length λ needs to connect the points $(L, 0)$ and $(0, 2L)$. Find the path of the string that minimizes the potential at the origin. *Hint*: start by finding a polar curve $\phi(\rho)$, and rewrite it as $\rho(\phi)$ after you find it.

14.60 **The Beltrami Identity** It can be shown that the Euler-Lagrange equation is equivalent to the equation $\partial f/\partial x - d/dx[f - y'(\partial f/\partial y')] = 0$. This form is generally less useful for solving problems, but when f doesn't depend on x this simplifies to $f - y'(\partial f/\partial y') = C$, which is known as the "Beltrami identity." In Section 14.2 we derived the equation for the shortest distance between two points on a 45° cone. In this problem you'll redo that calculation using the Beltrami identity. The starting point was the distance formula: $ds = \sqrt{2dz^2 + z^2 d\phi^2}$.

(a) In Section 14.2 we factored out dz to write an integral in terms of a function $\phi(z)$. Why did we choose to do it that way instead of writing it in terms of $z(\phi)$?

(b) This time factor out a $d\phi$ and write an integral for the distance s in terms of the function $z(\phi)$.

(c) Use the Beltrami identity to write a differential equation for $z(\phi)$.

(d) You could solve this equation with separation of variables, but the integral turns out to be a pain. Instead, plug in the solution we already found for $z(\phi)$ and verify that it solves the differential equation you wrote in Part (c).

14.61 Suppose you were asked to find stationary solutions to $\int \left(y^2 + x^2 y'\right) dx$.

(a) Apply the Euler-Lagrange equation in the usual way. What equation do you end up with? You should find that you get an exact answer with no arbitrary constants, even though we didn't specify boundary conditions.

(b) Suppose you wanted to minimize this particular integral between the points $(0, 0)$ and $(1, 1)$. What does your solution to Part (a) imply about the best curve to choose?

(c) Suppose you wanted to minimize this particular integral between the points $(0, 0)$ and $(1, 2)$. What does your solution to Part (a) imply about the best curve to choose?