Motivating Exercise for Partial Derivatives: The Wave Equation

A string that can vibrate up and down is described by a height function $y(x, t)$. The drawing below shows a string at some instant that we'll call $t = 10$, and labels a point indicating that $y(4, 10) = 2$. Assume throughout this exercise (except Part 4) that the ends of the string at $x = 0$ and $x = 10$ are tied down, meaning $y = 0$ at those points.

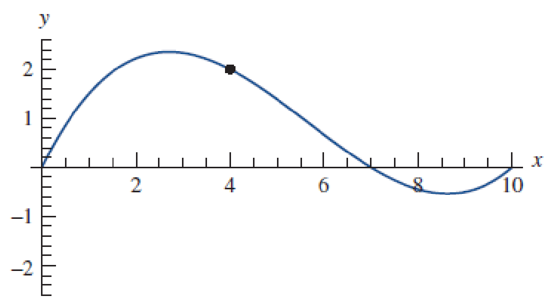

Figure 1: A vibrating string

1. The slope of the string at the point labeled above is "the derivative of y with respect to x," sometimes written $\partial y/\partial x$. Estimate $\partial y/\partial x$ at the point shown by looking at the drawing.

2. You cannot estimate "the derivative of y with respect to t" by looking at the drawing, but suppose we told you that at this particular point on the string at this particular moment, $\partial y/\partial t = -3$. Estimate where that piece of string would be at $t = 10.5$.

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An "equation of motion" relates the acceleration of an object to its position and/or velocity. For example, a mass attached to a rusted spring might obey the equation $d^2x/dt^2 = -(b/m)(dx/dt) - (k/m)x$. If you know the position and velocity at any given moment, you can use this equation to calculate the acceleration and thereby predict the motion over time.

The equation of motion for our string is the "wave equation":

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

On the left is the second derivative of y with respect to t: the vertical acceleration. On the right is the second derivative of y with respect to x: the concavity. This equation asserts that if you know the concavity at any given point—any particular x and t—you can determine the acceleration at that point. If you know the shape of the entire string at a particular moment, as well as the velocity at each point, you can use this equation to predict the motion of the string over time.

3. Consider a string that begins at rest stretched into a horizontal line. (We can express this mathematically by writing $y(x, 0) = 0$.) Based on Equation 1, what will happen to this string over time? Your answer will not require calculations, but you shouldn't just say what you would expect physically. Explain what behavior Equation 1 predicts by thinking about concavity and acceleration.

4. Now consider a string that begins at rest stretched into a diagonal line from the point $(0, 0)$ to the point $(10, 20)$, so $y(x, 0) = 2x$. Based on Equation 1, what will happen to this string over time?

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5. Now consider the string whose initial position $y(x, 0)$ is represented in Figure 1. Based on Equation 1, which parts of the string will accelerate downward? Which parts will accelerate upward?